Coverage of Continuous Regions in Euclidean Space
Using Homogeneous Resources
With Application to the Allocation of
Phased Array Radar Systems

THESIS

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Abstract

Air surveillance of United States territory is an essential Department of Defense (DoD) function. In the event of an incoming aerial attack on North America such as a hijacked or enemy airplane, missile, or any other National Security threat, the DoD, Department of Homeland Security (DHS), and Federal Aviation Administration (FAA) surveillance capabilities are critical to discovering and tracking the threat so that it can be eliminated. Many of the currently used surveillance radar will reach the end of their design life within ten to twenty years. The current surveillance system has significant low altitude surveillance gaps and limited ability to detect small radar cross section objects such as small missiles. By replacing the current radar network with a single integrated network of Multifunction Phased Array Radar (MPAR) units, surveillance capabilities can be enhanced and life cycle cost can be reduced. The problem of determining the location and number of required MPAR units to provide sufficient air surveillance of a given area is a large problem that could require a prohibitively long time to solve. The method used to solve this problem must be capable of handling changes to the system such as changes to MPAR capabilities or surveillance area. By representing the area of surveillance as a polygon and the MPAR units as guards with a defined circle of detection, this problem as well as other similar surveillance or coverage problems can be expressed with easily adjustable parameters.

The problem of covering the interior and exterior of a polygon region with a minimal number of guards with homogeneous capabilities is not well researched. There are no methods for determining the minimal number of guards required to cover the interior and exterior of a polygon at a desired coverage level less than 100 percent. This paper describes an iterative method for determining a small number and location of guards required to cover a convex polygon both fully and at a specified percentage coverage less than 100 percent. Analysis of test cases compared with other papers are presented. Specifically, results are presented to show that the developed methodology produces a smaller number of required MPAR units using less time than a comparable method presented in the literature. A goodness measure of the method is presented with respect to a lower bound for over 1000 test cases. Results for the United States Northern Command MPAR instance of this problem are presented to provide full and partial coverage of the Continental United States and 25 key cities of interest. The methodology developed in this thesis can be used to provide minimal cost surveillance recommendations over key areas or events, placement of communications resources, or other limited range resources.

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Thank you to my mom and dad whose support has been critical to my success. Without them and the rest of my amazing family I would not be where I am today- Thank you with all my heart.

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What a joy it is to continually learn new things...

Kassandra M Merritt

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List of Abbreviations

Abbreviation	Page
Department of Defense (DoD)	1
Department of Homeland Security (DHS)	1
Federal Aviation Administration (FAA)	1
National Airspace System (NAS)	1
Mean Sea Level (MSL)	1
Multi-Function Phased Array Radar (MPAR)	1
United States Northern Command (NORTHCOM)	2
Continental United States (CONUS)	2
Above Ground Level (AGL)	2
Area of Surveillance (AOS)	2
Nautical Miles (NM)	8
Observation Point (OP)	9
Mean Point (MP)	10
(Maximum Distance Vertex (MDV))	11
Median Observation Point with Adjusted Division (MOPAD)	19

Coverage of Continuous Regions in Euclidean Space Using Homogeneous Resources With Application to the Allocation of the Phased Array Radar Systems

1. Problem Statement and Overview

1.1 Problem Motivation

Air surveillance of United States territory is an essential Department of Defense (DoD) function. In the event of an incoming aerial attack on North America such as a hijacked or enemy airplane, missile, or any other National Security threat, the DoD, Department of Homeland Security (DHS), and Federal Aviation Administration (FAA) surveillance capabilities are critical to discovering and tracking the threat so that it can be eliminated. The current National Airspace System (NAS) provides coverage from the surface to 60,000 feet Mean Sea Level (MSL) using primary and secondary FAA long and short range radars, defense radars, and additional surveillance systems along the borders and other areas of interest [?]. The current radar system consists of weather and aerial surveillance radars that operate by using a rotating antenna to sweep a large area [?]. Many of these radars will reach the end of their design life within ten to twenty years [?]. The current surveillance system has significant low altitude surveillance gaps and limited ability to detect small radar cross section objects such as small missiles [?].

The Multi-Function Phased Array Radar (MPAR) has several mission capabilities including weather and aerial surveillance. A single MPAR unit is capable of tracking current weather conditions such as developing thunderstorms while also tracking numerous independently operating private and commercial aircraft. Current technology requires multiple radar units to independently track aircraft and weather. These MPAR units operate by directing an array of radar beams from a stationary surface instead of using a rotating antenna [?]. As a result of the reduced number of moving parts, the MPAR have increased reliability over traditional radar technology [?]. By replacing the current radar network with a single integrated network of MPAR units, surveillance capabilities can be greatly enhanced and life cycle cost can be reduced[?]. These capability improvements as well as increased reliability support national objectives outlined in the "Recommendations for Development and Implementation of Surveillance Capabilities in Support of the National Strategy for Aviation Security" [?]. This document specifies a desired outcome of a "Fully integrated, low medium and high altitude surveillance coverage with seamless network integration that leverages the full range of inter-agency sensor systems, capabilities, and analytic support tools to detect, monitor and track airborne objects with the National Airspace System [?]"

Due to the current coverage gaps and age of the NAS, United States Northern Command (NORTHCOM) is investigating upgrading the NAS to use MPAR technology. NORTHCOM is interested in determining the minimum number and location of units required to attain a given percent coverage of the Continental United States (CONUS) at different altitudes. Percent coverage is defined as the percent of area over which coverage is desired within range of the MPAR units. Specifically, the research sponsor, NORTHCOM J84, wants to determine the minimal number and location of MPAR units required to achieve 100 percent coverage of CONUS, at 500 feet Above Ground Level (AGL). The sponsor is interested in determining the number and location of MPAR units required to cover 25 key cities of interest within CONUS, given a radius around these points at varying altitudes and percent coverages.

1.2 Problem Statement

Given an enclosed Area of Surveillance (AOS) and a set of resources(such as guards, radars, cameras, security personnel, etc.) capable of seeing a set distance, range, in all directions (360 degrees), the problem is to determine the smallest number of these resources required to cover the entire area. Any point in the AOS is considered covered if it is within the defined range of at least one resource. Due to other possible restrictions, it is also important to determine the smallest number of resources required to partially cover the area at a specified percentage less than 100. The purpose of this research is to develop a robust methodology for determining a small number and location of resources (guards) with limited visibility range to cover a given AOS. The developed methodology is applicable to problems related to surveillance over key areas or events, the placement of key communications resources, or other limited range resources.

1.2.1 NORTHCOM Application. In the NORTHCOM instance of this problem, each MPAR is a guard with visibility range limited to line of sight detection range of the MPAR unit and the AOS is CONUS. The developed methodology is robust enough to handle system constraint changes such as range or altitude of detection, desired AOS, and percent coverage.

1.2.2 Research Questions. This research seeks to determine if there are any methods to quickly determine the smallest number and location of guards required to cover a given AOS completely or at a specified percentage less than 100 percent. This research seeks to develop a method to improve existing methods by providing a more minimal solution using less computation time. Additionally, this research seeks to determine how well the developed methodology performs as compared to existing methods and as compared to a lower bound.

1.2.3 Research Scope. This research provides a methodology for determining a small set of guards required to cover a given AOS. The methodology is capable of covering the entire area or a specified percentage (less than 100) of the area. The coded heuristic is provided as well as computation results for a variety of test scenarios. Numerical results and computational time requirements are compared with other continuous methods from the literature. A performance bound is discussed for the developed methodology. Numerical results are also presented for the NORTHCOM instance of this problem presented in Section **??**.

1.3 Research Contribution

This research provides a review of available literature pertaining to covering a continuous region with a small set of limited visibility guards. The applicability and shortcomings of available methods are discussed. This research develops a methodology to fill the current literature gap and provide a methodology capable of determining a small set of guards for covering a given AOS at a specified level of coverage. The effectiveness of this methodology as compared to the numerical results and computational time of given methods is presented and a performance measure of the method is provided.

1.4 Overview

1.4.1 Chapter 2: Literature Review. Chapter 2 reviews available literature concerning similar problems. The applicability of several different models is discussed as well as a review of papers that motivated the methodology further developed in Chapter 3.

1.4.2 Chapter 3: Methodology. Chapter 3 outlines constraints and assumptions of the proposed methodology. An iterative method for determining a small set of guards required to cover a given AOS completely and at a percentage less than 100 is presented (i.e. partial coverage). Performance results are also presented.

1.4.3 Chapter 4: Analysis and Results. Chapter 4 reviews analysis of the presented methodology. This analysis includes results for several full coverage test cases as compared with other methodologies found in the literature. A performance prediction of the methodology as compared to a provable bound is presented. Empirical results are used to show the partial coverage methodology offered produces a coverage level at or above the specified level. Results and analysis of the NORHTCOM instance of the problem are presented for several specific coverage regions and coverage levels.

1.4.4 Chapter 5: Conclusions. Chapter 5 discusses conclusions resulting from the analysis of the methodology as well as the NORTHCOM problem. Future work and additional applications are also discussed.

2. Literature Review

2.1 Introduction

The question of how to optimally use a limited number of resources is not new. Researchers have tried to answer this question in several different forms. Many of these traditional problem models are limited in scope and or scale and not able to handle NORTHCOM's specific problem. This chapter outlines specific previous research as well as existing methodologies that have been developed to determine the minimal number and location of resources required to cover a defined area. This chapter also discusses the limitations of existing research and why a new methodology is required to fully answer NORTHCOM's questions.

2.2 Definitions

A polygon, P, is typically defined as a set of ordered points $p_1, ..., p_n \in \Re^2$, $p_i = (x_i, y_i)$, $n \geq 3$ called vertices and the edges defined by the line segments joining adjacent points and point p_n to point p_1 [?, ?]. The polygon, P is said to be a simple polygon if none of the non-consecutive edges of P intersect [?, ?]. For the purpose of this research the term polygon will be used to refer to the simple polygon, P, as defined above along with its interior. A convex polygon is a simple polygon in which a line segment drawn between any two points inside the polygon is completely contained in the polygon [?]. A point $p \in P$ is said to be visible from $q \in P$ if the line segment between p and q does not intersect the exterior of P [?]. A set C of points in P is said to illuminate, guard, or cover P if every point in P is visible from at least one point in C [?].

A diagonal of P is a line segment joining two nonadjacent vertices of P that does not intersect an edge of P [?]. The triangulation of P is the decomposition of P into triangles formed only from the edges and diagonals of P [?]. Any simple polygon P that contains n vertices can be decomposed into $n-2$ triangles [?]. Triangulation of polygons plays a central role in efficiently solving some resource location problems and many algorithms have been studied to efficiently triangulate polygons [?].

2.3 Facility Location Problems

Facility Location Problems seek to determine the optimal location for a set of facilities [?]. Modeled as integer programming problems, facility location problems seek to determine the minimal cost set of facilities capable of achieving demand at a set of points [?]. Facility location problems find the optimal set of n new facilities chosen from a possible set of $m > n$ sites such that the distance or cost between these new facilities and r existing facilities is minimized [?]. The NORTHCOM problem can be modeled as a facility location problem by discretizing the AOS to establish a discrete set of possible locations for each MPAR unit as well as locations that must be covered by the units. This application fits the NORTHCOM problem if the research was restricted to using existing radar sites or federally owned land only. It would also be applicable if only interested in surveilling a discrete set of points. Instead, this research seeks to cover a continuous area rather than a set of discrete locations within that area and assumes no limitations on possible facility locations. Because of the limited set of possible resource locations and desired coverage points required for Facility Location Problems, the NORTHCOM problem can more accurately be represented by a model that accounts for the assumption of no limitations on possible facility locations and the continuous AOS.

Facility Location Problems are difficult problems and therefore can have prohibitively long computational times for solving large problems [?]. As a result of the size of the AOS being significantly larger than each sensor's radius of coverage, the number of units required to achieve complete coverage is expected to be large. According to the World Atlas, the area of the CONUS is 2,959,062 square miles [?]. Each individual MPAR can cover an area of 5026.55 square miles at 500 ft AGL. A lower bound assuming zero coverage overlap requires over 588 MPAR units. This number is a lower bound on the problem and would require the shapes of coverage to match perfectly side-by-side for each unit, which is not possible for a circular coverage area of the MPAR. Consequently, the computational time associated with solving large-scale Facility Location Problems could be prohibitive for solving a problem as large as the NORTHCOM one.

2.4 Art Gallery Problems

The Art Gallery problem, first addressed by Victor Klee in 1973 seeks to determine the minimal number of guards necessary and sufficient to see the walls of an art gallery represented as a 2D space [**?**, **?**, **?**, **?**]. Classically, the art gallery is a rectangular shape in the 2D plane with walls dividing the area into smaller rectangles [**?**]. The guards in the classic art gallery problem have 360 degree view with unlimited visibility range [**?**]. More recent studies have expanded this definition to include simple and orthogonal polygons with and without holes as well as limited visibility directions such as search light problems [**?**]. Art Gallery problems seek to determine the minimal cardinality set C such that all points in P are visible [**?**]. The NORTHCOM problem can be represented as an art gallery problem where the AOS is the polygon and MPAR units are the guards with limited visibility range.

2.5 Limited Visibility Problems

Defining illumination in the traditional way, as discussed above, assumes guards have unlimited visibility range. This is not always a valid assumption. In the NORTHCOM problem the MPAR units' range is assumed to be constant and limited. The visibility range of each MPAR unit is a characteristic dependent on the altitude of the object being detected. For example, if an object, such as a plane is at an altitude of 500 feet AGL, the MPAR unit is capable of detecting this object at 40 Nautical Miles (NM) from the MPAR. If, however the obect is at an altitude of 20,000 feet AGL, the object could be detected at a range of 186 NM. The visibility range of the MPAR is dependent on the desired altitude of coverage. The visibility range is calculated for a given altitude assuming line-of-site visibility for the MPAR. An MPAR unit has a straight line of visibility while the Earth's surface is round. Due to functional requirements, MPAR units are built at a height of 100 feet AGL. Based on this height and the desired altitude of coverage, a visibility range can be derived as a result of the inherent curvature of the earth. At higher altitudes, objects can be detected farther away. Consequently, a higher altitude of coverage equates to a larger visibility range of the MPAR. Figure ?? depicts how this visibility range is estimated based on the affect of the curvature of the earth.

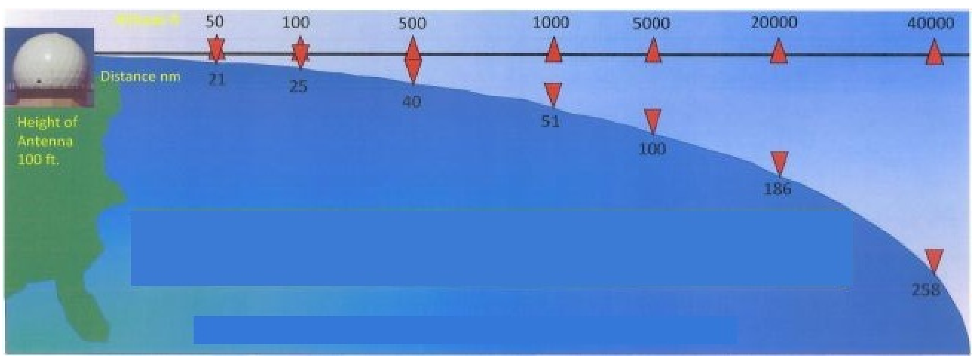


Figure 1 Range Calculation for MPAR Units

In 2002 Giorgos Kazazakis and Antonis Argyros of the Institute of Computer Science, Foundation for Research and Technology-Hellas in Heraklion, Crete, Greece published a paper titled "Fast Positioning of Limited-Visibility Guards for the Inspection of 2D Workspaces" in the Proceedings of IEEE [?]. This article develops a methodology to efficiently determine a small number of guards required to cover the edges of a polygon given limited visibility range of the guards [?]. The authors develop a methodology that decomposes the initial polygon, P, into convex sub-polygons [?]. A potential Observation Point (OP) is determined and the question is asked "can all points in the current sub-polygon be [covered] by this point?" [?]. If all points in the current sub-polygon can be covered by the OP then the OP is added to the set of valid OPs in the solution and the sub-polygon is considered covered

[?]. If not, the polygon is divided into more sub-polygons. This is repeated until all points on the exterior of the original polygon are within illumination range of the set of valid OPs [?].

According to Kazazakis and Argyros, the optimal OP "is the center of the minimum-radius circle that contains the polygon" [?]. However, the selection of the OP must be computationally inexpensive because the OP is recalculated at every iteration. A common way to determine the OP is based on the Mean Point (MP) of the polygon (Equation (??)) but this method produces an OP that will result in more required guards in polygons with a long tail, such as the one shown in Figure ?? [?]. Instead, using an observation point based on the median of the polygon, as calculated by Equation (??), will result in a smaller number of required guards because the selected point will be biased towards the long edge of the polygon [?]. Kazazakis and Argyros present experimental results to demonstrate the use of median point as the OP results in a smaller number of required guards than using the MP as the OP [?].

$$MP = \frac{\sum_{i=1}^{n} p_i}{n} \tag{1}$$

$$OP = \sum_{i=1}^{n} \|E_i\| M_i / \sum_{i=1}^{n} \|E_i\| \tag{2}$$

$$M_i = \text{coordinates of midpoint of } i\text{th edge, } E_i \text{ of polygon } P \tag{3}$$

$$\|E_i\| = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \text{ for } i = 1, 2, ..., n-1 \tag{4}$$

$$\|E_n\| = \sqrt{(x_n - x_1)^2 + (y_n - y_1)^2} \tag{5}$$

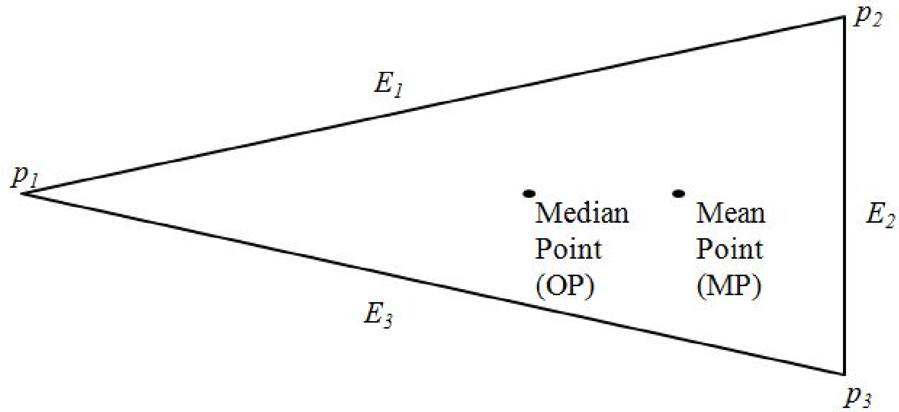
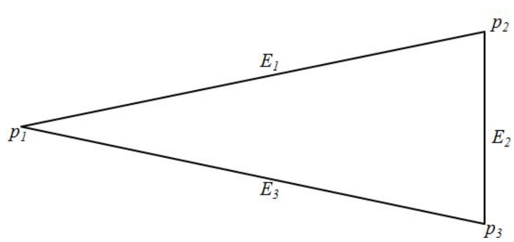
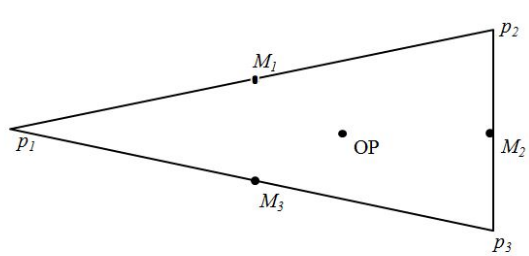


Figure 2 Mean Point (MP) vs Median Point (OP)

After determining a potential OP, as shown in Figure ??, the Euclidean distance between each vertex of the polygon and the OP is calculated [?]. Equation (??) shows how the distance from vertex i to the OP is calculated [?]. If the vertex with the maximum distance from the OP (Maximum Distance Vertex (MDV)) is within range of the guard (Figure ??(a)) then the entire polygon is covered and the OP should be used [?]. Otherwise (see Figure ??(b)), the polygon is not covered from the single OP and should be divided into sub-polygons [?].

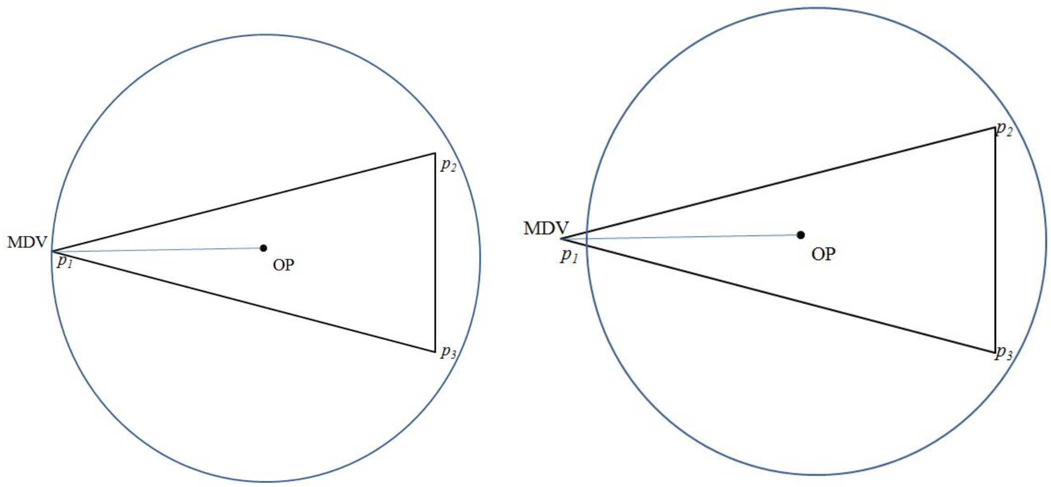


(a) Polygon



(b) OP Calculation

Figure 3 Selecting an OP



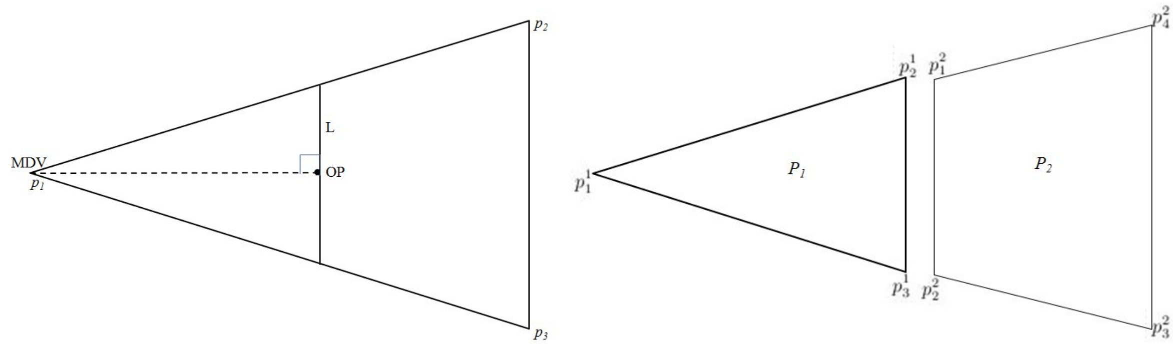
(a) Covered Polygon (b) Not Covered Polygon

Figure 4 Determining if a Polygon is Covered

$$d_i = \sqrt{(x_i - x_{OP})^2 + (y_i - y_{OP})^2} \tag{6}$$

A polygon not covered by the selected OP is divided into sub-polygons inspected using the same procedure as the original polygon [?]. Since each sub-polygon requires a separate OP to cover, it is important to divide the polygon into as few sub-polygons as required to ensure a minimal number of required OPs [?]. The line used to divide the polygon should decrease the distance of the OP from the MDV in the new sub-polygons as much as possible [?]. Using the line, L, defined by the line perpendicular to the line between the MDV and OP and passing through the OP will achieve this objective [?]. This line is shown in Figure **??** (a) as L and the resultant sub-polygons are shown in Figure **??** (b).

An example of Kazazakis and Argyros' method applied to a convex polygon is shown in Figure **??**. In this figure the line at the bottom right shows the visibility range of the guards [?]. Line 1 is first chosen to divide the initial polygon [?]. When



(a) Choice of L (b) Sub-polygons

Figure 5 Deviding a Nonvisible Polygon into Sub-polygons

the sub-polygons are not entirely covered they are divided again [?]. The final result is shown with selected OPs shown as red dots inside the sub-polygons [?]. It can be seen in this figure that the two triangles formed by lines 1,2, and 3 do not have selected OPs. This is because Kazazakis and Argyros' method is used to determine a small set of guards required to guard the exterior of the original polygon [?]. According to Kazazakis and Argyros, coverage of the entire polygon, including the interior, can be achieved by inspecting all of the sub-polygons instead of just those with vertices on the exterior of the original polygon [?].

In the article, "Covering a Compact Polygonal Set by Identical Circles", Stoyan and Patsuk discuss methods for covering a compact polygonal set with identical circles of minimal radius [?]. Stoyan and Patsuk discuss finding the minimum visibility range of a given number of guards to cover a polygon [?]. In this paper Stoyan and Patsuk develop a method for testing if the polygon is covered based on a calculated value that represents a measure for the uncovered area in P [?]. They present computational results for covering a 100 by 100 square as well as several polygons with different sizes and numbers of homogeneous guards [?].

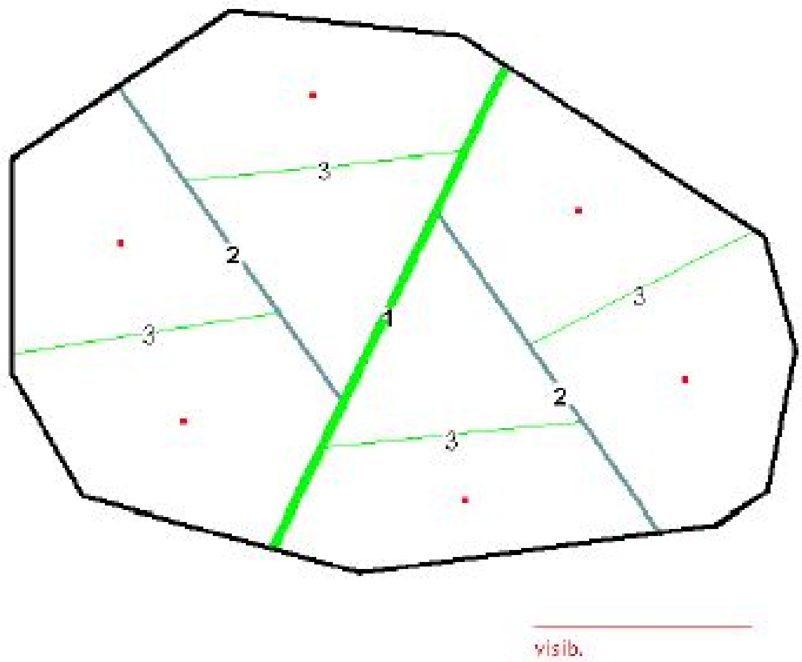


Figure 6 Example of Kazazakis and Argyros' Method [?]

2.6 Conclusion

Current literature is lacking in methods for covering the interior and exterior of a continuous region with limited visibility guards. Discrete techniques available for similar problems do not account for key assumptions necessary to the NORTHCOM problem such as a continuous AOS and may have prohibitively long computational times. While Kazazakis and Argyros present a method for limited visibility coverage of the exterior of a polygon and suggest the same method for covering the interior of the polygon under the same conditions, little computational results or proof of concept are presented. The literature does not present a method for covering a given percentage, less than 100, of a polygonal region under limited visibility. Due to functional or financial restrictions placed on the real world problems for which these methods are applicable, the capability to look at less than full coverage would be beneficial. This research seeks to fill this literature gap by developing a methodology that can be used to quickly determine a small set of limited visibility guards to cover the interior and exterior of a polygon at a specified level of coverage less than or equal to 100 percent.

3. Methodology

3.1 Overview

Current literature is lacking in methods for covering the interior and exterior of a continuous region with limited visibility guards. While Kazazakis and Argyros [?] present a method for limited visibility coverage of the exterior of a polygon and suggest the same method for covering the interior of the polygon under the same conditions, little computational results or proof of concept are presented. The literature does not present a method for covering a given percentage, less than 100, of a polygonal region under limited visibility. Due to functional or financial restrictions placed on the real world problems for which these methods are applicable, the capability to look at less than full coverage is beneficial. The full coverage method developed in this section is based on Kazazakis and Argyros' method [?] discussed in Chapter 2 with modifications to the selection of cut line. The partial coverage methodology developed in this section is similarly based on Kazazakis and Argyros' method [?], however uses a measure for the uncovered portion of the region inspired by Stoyan and Patsuk's ideas as presented in Chapter 2.

3.2 Definition of Terms

Area of Surveillance (AOS): The territory, defined by user input, over which surveillance is required. This area is defined by latitude and longitudinal coordinates describing the vertices of the area.

Guard: Resource providing coverage. For the NORTHCOM scenario each MPAR unit is a guard.

Observation Point (OP): Location to station a guard as calculated by Equation (??)

Circle of Detection: Circle around a guard such that any point within this circle is covered.

Visibility Range (β): Range from OP that an object is covered by the guard. β is expressed as a single number representing the radius of the circle detection of the guard

3.3 Mathematical Formulation

3.3.1 Inputs.

$P = [p_1, p_2, ..., p_n]$ where $p_i = (x_i, y_i) \in \Re^{2+}$ is the location of vertex i for the convex polygon P.

$\beta =$ visibility range of guard given in the same basic unit of measure as the coordinate system for P.

$\delta =$ level of detection required, expressed as a percentage such that $0 < \delta \leq 100$.

3.3.2 Outputs.

$Z = z_1, z_2, ..., z_r$ where $z_j = (x_{OP_j}, y_{OP_j})$

3.3.3 Assumptions. In order to determine a small set of guards required to cover a given polygon several assumptions are made. The polygon, P is convex and contained in the positive quadrant of the x, y plane. The set of guards have homogeneous capabilities with visibility range β. The visibility of each guard is constant and 360 degrees. The location of guards is constrained to P.

3.4 Overview of Formulation

Because of the assumptions stated in Section **??**, this problem is similar to the problem studied by Kazazakis and Argyros [**?**]. While the areas of coverage for each sensor are circles of given radius, β, and P is a simple polygon as in Kazazakis and Argyros' problem, the additional questions related to different percent coverage, δ, and coverage of the interior of P differ from Kazazakis and Argyros' problem

[?]. Because of the similar structure, the developed method decomposes the given polygon P into sub-polygons using a method similar to Kazazakis and Argyros' [?] and similarly checks for 100 percent coverage by checking for coverage of the vertex farthest away from the OP. However, the developed methodology also considers the interior of P and uses a different cut line to divide the polygon into sub-polygons. The methodology is also expanded to account for incidences of less than 100 percent required coverage.

3.5 Median Observation Point with Adjusted Division (MOPAD) Methodology

Given inputs of convex polygon P, visibility range β, and percent coverage $0 < \delta \leq 100$, a potential OP is calculated based on Kazazakis and Argyros' method, that is, the median point of the polygon is calculated using Equations (**??**)-(**??**) [?]. If P is covered at a level $\geq \delta$ using the selected OP (see Sections **??** and **??**) then the OP is a valid point. If P is not covered at a level $\geq \delta$ (see Sections **??** and **??**) then the OP is not valid and P must be divided into sub-polygons.

$$OP = \sum_{i=1}^{n} \|E_i\| M_i / \sum_{i=1}^{n} \|E_i\| \tag{7}$$

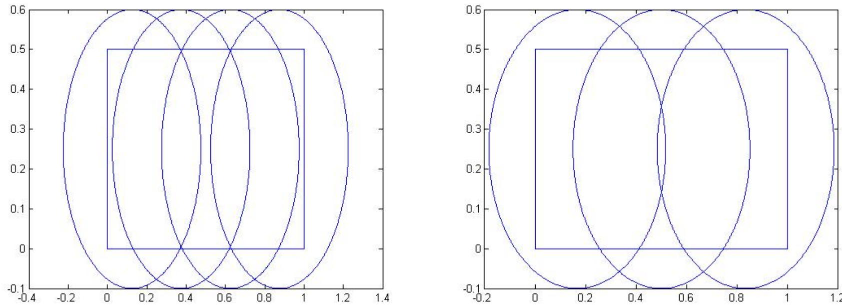
$$\text{Where } M_i = \text{coordinates of midpoint of } i\text{th edge, } E_i \text{ of polygon } P \tag{8}$$

$$\|E_i\| = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \text{ for } i = 1, 2, ..., n-1 \tag{9}$$

$$\|E_n\| = \sqrt{(x_n - x_1)^2 + (y_n - y_1)^2} \tag{10}$$

Kazazakis and Argyros divide P into sub-polygons using the line through the OP that is perpendicular to the line between the OP and the MDV (shown as L in Figure **??** (a)) [?]. Further assessment of this method (shown in Chapter 4) shows that the results are not as good as expected. Consequently, a new method for dividing P into sub-polygons was developed. Stoyan and Patsuk [?] present consistently more minimal numerical results than Kazazakis and Argyros' [?] but do not specify how they selected locations for the guards (see Chapter 4). However, the numerical results presented by Stoyan and Patsuk show that the locations of the guards are often in a grid-like pattern [?]. These results motivated the idea of dividing P using only vertical or horizontal lines in order to maintain a more grid-like structure for the locations.

The horizontal and vertical distances between the OP and the MDV, x_{dist} and y_{dist}, are calculated (see Equation (**??**) and (**??**) respectively). If the horizontal distance is larger than the vertical distance then P is divided using a horizontal line. If the horizontal distance is smaller than the vertical distance, P is divided using a vertical line. Three implementations of the Median Observation Point with Adjusted Division (MOPAD) method are presented. The first, MOPAD1, divides P using horizontal and vertical lines that pass through the OP. There are some instances where three guards could sufficiently cover the width or height of P but two guards cannot. For example, consider $P = [0, 0;\ 1, 0;\ 1, 0.5;\ 0, 0.5;\ 0, 0]$ and $\beta = .35$ (Figure **??**). If only MOPAD 1 is implemented, these instances will result in 4 guards being used instead of three because the MOPAD1 method will divide P in half and then each of those sub-polygons in half again. In these instances, dividing P in thirds will result in a smaller number of required guards. In order to account for these instances, MOPAD2 and MOPAD3 were developed. The second implementation, MOPAD2, divides P into thirds instead of halves if the x_{dist} or y_{dist} is greater than 2β but less than 4β and in half otherwise. The third implementation, MOPAD3, divides P into thirds using a vertical or horizontal line if the x_{dist} or y_{dist} is between $(1.5)\beta$ and $(2.5)\beta$ and in half otherwise. To find the best solution, all three implementations of the MOPAD methodology are run on a given problem and the best solution is used. Best is defined as the smallest number of guards needed to provide full coverage.



(a) MOPAD1 Implemented on P (b) MOPAD2 or MOPAD3 Implemented on P

Figure 7 MOPAD1 vs MOPAD2 and MOPAD3

$$x_{dist} = |x_{MDV} - x_{OP}| \tag{11}$$

$$y_{dist} = |y_{MDV} - y_{OP}| \tag{12}$$

3.5.1 Checking Visibility for Full Coverage. Given $\delta = 100$, the distance between the MDV and the OP, denoted as α is calculated as shown in Equation (**??**) [**?**]. If $\alpha \leq \beta$ then P is covered and the OP is considered a valid point. If $\alpha > \beta$ then the OP is not valid and P must be divided into sub-polygons as described in Section **??**.

$$\alpha = \max_{i=1,\ldots,n} \sqrt{(x_i - x_{OP})^2 + (y_i - y_{OP})^2} \tag{13}$$

3.5.2 Checking Visibility for Partial Coverage. To determine if P is covered at a level greater than or equal to $0 < \delta < 100$, an estimate of the percent covered portion of P, κ is calculated and compared to the required level of coverage. This concept is motivated by a similar measure used in Stoyan and Patsuk's paper to estimate the uncovered portion of P [**?**]. The distance between each vertex of P and the OP is calculated as shown in Equation (**??**). The value κ is calculated using Equation (**??**). κ is then compared to λ, as calculated in Equation (**??**). For more information on how this relationship was determined see Section **??**.

$$dist_i = \sqrt{(x_i - x_{OP})^2 + (y_i - y_{OP})^2} \tag{14}$$

$$\kappa = \frac{\beta^2}{\sum_{i=1}^{n} dist_i} \tag{15}$$

$$\lambda = \frac{\delta}{n} \tag{16}$$

If κ, the estimate of the percent coverage of P, is greater than or equal to λ, a measure of the required coverage, then P is covered at a level greater than or equal to δ and the OP is considered a valid point. If $\kappa < \lambda$ then the OP is not valid and P must be divided into sub-polygons using the method described in Section **??**.

3.6 Pseudocode

3.6.1 MOPAD1 for Full Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E_i^j\| = \sqrt{(x_i^j - x_{i+1}^j)^2 + (y_i^j - y_{i+1}^j)^2}$ for $i = 1, ..., (n-1)$

$\|E_n^j\| = \sqrt{(x_n^j - x_1^j)^2 + (y_n^j - y_1^j)^2}$

M_i^j = coordinates of midpoint of ith edge, E_i^j , of polygon P_j

$OP_j = \sum_{i=1}^{n} \|E_i^j\| M_i^j / \sum_{i=1}^{n} \|E_i^j\|$

(Calculate α)

$d_i^j = \sqrt{(x_i^j - x_{OP}^j)^2 + (y_i^j - y_{OP}^j)^2}$

$\alpha^j = \max_{i=1,\ldots,n} d_i^j$

MDV^j = index of MDV

(Test for coverage)

IF $\alpha^j \leq \beta$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE (If P_j is not covered)

$x_{dist}^j = |x_{MDV^j}^j - x_{OP}^j|$

$y_{dist}^j = |y_{MDV^j}^j - y_{OP}^j|$

IF $|x_{dist}^j| \geq |y_{dist}^j|$

A_{n+1} and A_{n+2} are the sub-polygons formed by cutting P_j vertically through t

ELSE

A_{n+1} and A_{n+2} are the sub-polygons formed by cutting P_j horizontally throug

$j = j + 1$

END

3.6.2 MOPAD2 for Full Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E_i^j\| = \sqrt{(x_i^j - x_{i+1}^j)^2 + (y_i^j - y_{i+1}^j)^2}$ for $i = 1, ..., (n-1)$

$\|E_n^j\| = \sqrt{(x_n^j - x_1^j)^2 + (y_n^j - y_1^j)^2}$

M_i^j = coordinates of midpoint of ith edge, E_i^j , of polygon P_j

$OP_j = \sum_{i=1}^{n} \|E_i^j\| M_i^j / \sum_{i=1}^{n} \|E_i^j\|$

(Calculate α)

$d_i^j = \sqrt{(x_i^j - x_{OP}^j)^2 + (y_i^j - y_{OP}^j)^2}$

$\alpha^j = \max_{i=1,\ldots,n} d_i^j$

MDV^j = index of MDV

(Test for coverage)

IF $\alpha^j \leq \beta$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE

$x_{dist}^j = |x_{MDV^j}^j - x_{OP}^j|$

$y_{dist}^j = |y_{MDV^j}^j - y_{OP}^j|$

IF $|x_{dist}^j| \geq |y_{dist}^j|$

IF $|x_{dist}^j| \geq 2\beta$ and $|x_{dist}^j| \leq 4\beta$

P_{n+1}, P_{n+2} and P_{n+3} are the sub-polygons formed by cutting P_j in thirds vertically

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j vertically through the OP

ELSE

IF $|y_{dist}^j| \geq 2\beta$ and $|y_{dist}^j| \leq 4\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j in thirds horizontally

ELSE

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j horizontally through the OP

$j = j + 1$

END

3.6.3 MOPAD3 for Full Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E_i^j\| = \sqrt{(x_i^j - x_{i+1}^j)^2 + (y_i^j - y_{i+1}^j)^2}$ for $i = 1, ..., (n-1)$

$\|E_n^j\| = \sqrt{(x_n^j - x_1^j)^2 + (y_n^j - y_1^j)^2}$

M_i^j = coordinates of midpoint of ith edge, E_i^j , of polygon P_j

$OP_j = \sum_{i=1}^n \|E_i^j\| M_i^j / \sum_{i=1}^n \|E_i^j\|$

(Calculate α)

$d_i^j = \sqrt{(x_i^j - x_{OP}^j)^2 + (y_i^j - y_{OP}^j)^2}$

$\alpha^j = \max_{i=1,\ldots,n} d_i^j$

MDV^j = index of MDV

(Test for coverage)

IF $\alpha^j \leq \beta$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE

$x_{dist}^j = |x_{MDV^j}^j - x_{OP}^j|$

$y_{dist}^j = |y_{MDV^j}^j - y_{OP}^j|$

IF $|x_{dist}^j| \geq |y_{dist}^j|$

IF $|x_{dist}^j| \geq 1.5\beta$ and $|x_{dist}^j| \leq 2.5\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j

in thirds vertically

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j

vertically through the OP

ELSE

IF $|y^j_{dist}| \geq 1.5\beta$ and $|y^j_{dist}| \leq 2.5\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j

in thirds horizontally

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j

horizontally through the OP

$j = j + 1$

END

3.6.4 MOPAD1 for Partial Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E^j_i\| = \sqrt{(x^j_i - x^j_{i+1})^2 + (y^j_i - y^j_{i+1})^2}$ for $i = 1, ..., (n-1)$

$\|E^j_n\| = \sqrt{(x^j_n - x^j_1)^2 + (y^j_n - y^j_1)^2}$

M^j_i = coordinates of midpoint of ith edge, E^j_i , of polygon P_j

$OP_j = \sum_{i=1}^{n} \|E^j_i\| M^j_i / \sum_{i=1}^{n} \|E^j_i\|$

(Calculate $\lambda and \kappa$)

$d^j_i = \sqrt{(x^j_i - x^j_{OP})^2 + (y^j_i - y^j_{OP})^2}$

$\kappa^j = \frac{\beta^2}{\sum_{i=1}^{n} d^j_i}$

$\lambda = \frac{\delta}{n}$

MDV^j = index of MDV

(Test for coverage)

IF $\kappa^j \geq \lambda$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE

$x^j_{dist} = |x^j_{MDV^j} - x^j_{OP}|$

$y^j_{dist} = |y^j_{MDV^j} - y^j_{OP}|$

IF $|x^j_{dist}| \geq |y^j_{dist}|$

P_{j+1} and P_{j+2} are the sub-polygons formed by cutting P_j vertically through th

ELSE

P_{j+1} and P_{j+2} are the sub-polygons formed by cutting P_j horizontally through

$j = j + 1$

END

3.6.5 MOPAD2 for Partial Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E^j_i\| = \sqrt{(x^j_i - x^j_{i+1})^2 + (y^j_i - y^j_{i+1})^2}$ for $i = 1, ..., (n-1)$

$\|E^j_n\| = \sqrt{(x^j_n - x^j_1)^2 + (y^j_n - y^j_1)^2}$

M^j_i = coordinates of midpoint of ith edge, E^j_i , of polygon P_j

$OP_j = \sum_{i=1}^n \|E^j_i\| M^j_i / \sum_{i=1}^n \|E^j_i\|$

(Calculate $\lambda and \kappa$)

$d^j_i = \sqrt{(x^j_i - x^j_{OP})^2 + (y^j_i - y^j_{OP})^2}$

$\kappa^j = \frac{\beta^2}{\sum_{i=1}^{n} d_i^j}$

$\lambda = \frac{\delta}{n}$

MDV^j = index of MDV

(Test for coverage)

IF $\kappa^j \geq \lambda$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE

$x^j_{dist} = |x^j_{MDV^j} - x^j_{OP}|$

$y^j_{dist} = |y^j_{MDV^j} - y^j_{OP}|$

IF $|x^j_{dist}| \geq |y^j_{dist}|$

IF $|x^j_{dist}| \geq 2\beta$ and $|x^j_{dist}| \leq 4\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j vertically through the OP

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j in half vertically

ELSE

IF $|y^j_{dist}| \geq 2\beta$ and $|y^j_{dist}| \leq 4\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j in thirds horizontally

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j horizontally through the OP

$j = j + 1$

END

3.6.6 MOPAD3 for Partial Coverage.

$A\{1\} = P$

$n = size\{A\}$

$j = 0$

WHILE $j \leq n$: (while there are polygons or sub-polygons remaining that are not covered)

(Calculate the OP)

$\|E_i^j\| = \sqrt{(x_i^j - x_{i+1}^j)^2 + (y_i^j - y_{i+1}^j)^2}$ for $i = 1, ..., (n-1)$

$\|E_n^j\| = \sqrt{(x_n^j - x_1^j)^2 + (y_n^j - y_1^j)^2}$

M_i^j = coordinates of midpoint of ith edge, E_i^j , of polygon P_j

$OP_j = \sum_{i=1}^n \|E_i^j\| M_i^j / \sum_{i=1}^n \|E_i^j\|$

(Calculate $\lambda and \kappa$)

$d_i^j = \sqrt{(x_i^j - x_{OP}^j)^2 + (y_i^j - y_{OP}^j)^2}$

$\kappa^j = \frac{\beta^2}{\sum_{i=1}^n d_i^j}$

$\lambda = \frac{\delta}{n}$

MDV^j = index of MDV

(Test for coverage)

IF $\kappa^j \geq \lambda$: (if P_j is covered)

$m = size(Z)$

$Z_{m+1} = OP_j$ (Save OP in answer Set)

$j = j + 1$

ELSE

$x_{dist}^j = |x_{MDV^j}^j - x_{OP}^j|$

$y_{dist}^j = |y_{MDV^j}^j - y_{OP}^j|$

IF $|x_{dist}^j| \geq |y_{dist}^j|$

IF $|x_{dist}^j| \geq 1.5\beta$ and $|x_{dist}^j| \leq 2.5\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j in thirds vertically

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j

vertically through the OP

ELSE

IF $|y^j_{dist}| \geq 1.5\beta$ and $|y^j_{dist}| \leq 2.5\beta$

P_{j+1}, P_{j+2} and P_{j+3} are the sub-polygons formed by cutting P_j

in thirds horizontally

ELSE

P_{j+1}, P_{j+2} are the sub-polygons formed by cutting P_j

horizontally through the OP

$j = j + 1$

END

3.7 Adjusting for Violated Assumptions

The assumptions of a convex polygonal AOS in the positive x, y quadrant may not be applicable to every problem. The AOS can easily be adjusted to account for violated assumptions and allow for the use of the MOPAD methodology. If P is not a convex polygon, the convex hull of P, P^*, should be used as an input to the MOPAD method and any guards that cover regions of P^* that were not part of P should be manually removed from the solution. If P is not in the positive x, y quadrant, the entire region of P should be shifted to the positive x, y quadrant by adding a constant to each vertex. If the original AOS is not a polygon but instead some other shape, such as a circle, the original shape should be inscribed inside a polygon, P^*. The MOPAD method can then be run on P^* and all guards covering a region of P^* not originally included as part of the AOS can be manually removed from the solution.

3.8 Proof of Coverage Given Coverage of Sub-polygons

It can be proven that if each sub-polygon, P_i is covered at level $\delta_i \geq \delta$ then the entire polygon will be covered at a level greater than or equal to δ. Let C_{OP_i} be the circle centered at OP_i with radius β. Let δ^* be the coverage level of the entire polygon P, A_P be the area of polygon P, A_{P_i} be the area of sub-polygon P_i, A_{S_i} be the area of the region described by S_i, and A_{OP_i} be the area of C_{OP_i}. Then let $A_{OP_i \cap P_i}$ be the area contained in P_i that is covered by OP_i. This proof is applicable to all levels of coverage, $0 \leq \delta^* \leq 100$ percent.

$$\text{Let set } S_i = \bigcup_{j \neq i} [C_{OP_i} \cap (P_j \setminus C_{OP_j})] \ \forall i = 1, ..., n$$

$$\delta^* = \frac{\sum_{i=1}^n A_{OP_i \cap P_i}}{\sum_{i=1}^n A_{P_i}} + \sum_{i=1}^n A_{S_i}$$

$$\delta^* = \frac{\sum_{i=1}^n \delta_i A_{P_i}}{\sum_{i=1}^n A_{P_i}} + \sum_{i=1}^n A_{S_i}$$

$$\delta^* = \frac{\delta_1 A_{P_1} + \delta_2 A_{P_2} + ... + \delta_n A_{P_n}}{\sum_{i=1}^n A_{P_i}} + \sum_{i=1}^n A_{S_i}$$

$$\delta^* \geq \frac{\delta A_{P_1} + \delta A_{P_2} + ... + \delta A_{P_n}}{\sum_{i=1}^n A_{P_i}} + \sum_{i=1}^n A_{S_i} \text{ since } \delta_i \geq \delta \ \forall i = 1, ..., n$$

$$\delta^* \geq \frac{\delta \sum_{i=1}^n A_{P_i}}{\sum_{i=1}^n A_{P_i}} + \sum_{i=1}^n A_{S_i}$$

$$\delta^* \geq \delta + \sum_{i=1}^n A_{S_i}$$

$$\delta^* \geq \delta$$

3.9 Determining the Relationship between κ and λ

Let the inner circle in Figure **??** represent the circle of visibility of a guard at the center of the circle with range β_1. Let the area over which coverage is desired be the larger circle, with radius r_1. The percent coverage, δ_1, provided by the guard is

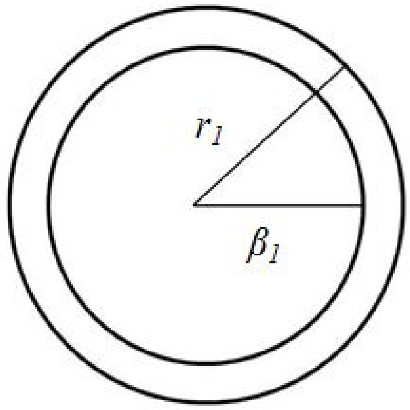


Figure 8 Setting a Bound on λ

given by $\delta_1 = \frac{\pi*\beta_1^2}{\pi*r_1^2} = \frac{\beta_1^2}{r_1^2}$. To insure that the area is covered at a level at least the desired percent coverage, δ, δ_1 must be greater than or equal to δ. In other words, $\frac{\beta_1^2}{r_1^2} \geq \delta$.

Any convex polygon can be placed inside the outer circle with the OP in the center of the smaller circle and r_i equal to the distance between the OP and each vertex. The area of this polygon will be smaller than the area of the outer circle regardless of the number of vertices, n, in the polygon. Therefore let δ be the bound on the ratio of β^2 and the sum of the distances of each vertex from the OP (See Equation (**??**)). This relationship can be proven for a circle and as shown is expected to extend to an inscribed polygon. Further testing (see Chapter 4) showed that the bound needed to be tightened and results were more successful by using the required relationship shown in Equation (**??**).

$$\frac{\beta^2}{\sum_{i=1}^{n}(r_i)} \geq \delta \tag{17}$$

$$\frac{\beta^2}{\sum_{i=1}^{n}(r_i)} \geq \frac{\delta}{n} \tag{18}$$

3.10 Conclusion

The MOPAD methodology developed in this chapter provides a method that quickly determines a small set of limited visibility guards capable of covering a polygonal region at a specified level of coverage less than or equal to 100 percent. The MOPAD method assumes a convex polygonal AOS contained in the positive x, y quadrant and 360 degree visibility guards with visibility range of β. There are available options for handling instances where these assumptions are validated. The MOPAD method improves upon the methodology presented by Kazazakis and Argyros [**?**] by using a different cut-line motivated by methods developed by Stoyan and Patsuk [**?**].

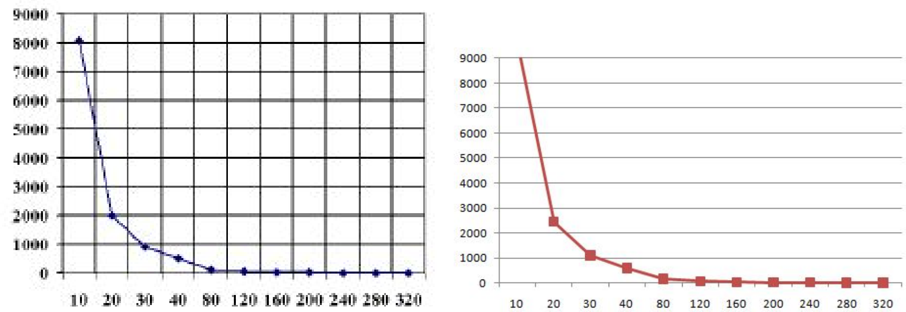
4. Analysis and Results

4.1 Overview

Current literature is lacking not only in available methods to solve the NORTHCOM problem but also in numerical results of these methods. In this chapter, the MOPAD method presented in chapter 3 is analyzed. The method is compared to Kazazakis and Argyros' method [?] and Stoyan and Patsuk's method [?] using examples presented in both papers. Computational results and computational time to solve are discussed. A performance prediction for the MOPAD method is made with respect to a lower bound. Empirical results are used to show the MOPAD method for partial coverage will provide a coverage level at least as high as requested. The NORTHCOM instance of the problem is solved for CONUS and 25 cities of interest to the sponsor.

4.2 Kazazakis and Argyros' Method

4.2.1 Verification and Validation. In Kazazakis and Argyros' paper, the number of guards required to cover a 1081 by 776 rectangle are shown with respect to the visibility range of the guards [?]. These results are shown in Figure **??**(a). The results of implementing Kazazakis and Argyros' methodology as presented in Chapter 2 in Matlab are shown in Figure **??**(b). It can be seen from these graphs that the methodology as implemented by Kazazakis and Argyros and the methodology as implemented by the author of this thesis present a similar trend in the required number of guards. While the exact numerical results are not presented by Kazazakis and Argyros, the results of implementation of Kazazakis and Argyros' method using Matlab appear to be at worst within around 20 percent of the values presented in Kazazakis and Argyros' paper [?]. It is possible that this difference is a result of different implementations. For example, it is possible to have a polygon or subpolygon that has a tie for which vertex is the MDV. This thesis arbitrarily breaks that tie by using the first vertex as the MDV. It is possible that Kazazakis and Argyros broke these ties using some other method and this may result in a slight variation to the results. For the purpose of the analysis presented in this paper the results referred to as Kazazakis and Argyros' method are produced using the implementation of Kazazakis and Argyros' method as presented in Chapter 2 and implemented in Matlab by the author of this thesis.



(a) Kazazakis and Argyros' Published Results [?]

(b) Author's Implementation of Kazazakis and Argyros' Method

Figure 9 Number of Guards as a Function of Visibility Range

4.2.2 Comparison to Stoyan and Patsuk. In the article "Covering a Compact Polygonal Set by Identical Circles", Stoyan and Putsuk present several numerical examples using their proposed methodology [?]. Given a 100 by 100 square and a given desired number of guards to cover that square, Stoyan and Patsuk find the minimum visibility range required for the guards to completely cover the square [?]. Using the same square and visibility range they determined as minimal for inputs to the Kazazakis and Argyros methodology, the number of guards required to cover the square are shown in column "K and A" in Apendix **??**. These results are also shown graphically in Figure **??**. While Stoyan and Patsuk were answering a different question, specifically what is the minimal visibility range required to achieve coverage with the given number of guards, these results show that on average Kazazakis and Argyros required 170 percent more guards with the same visibility range to cover the square than Stoyan and Patsuk.

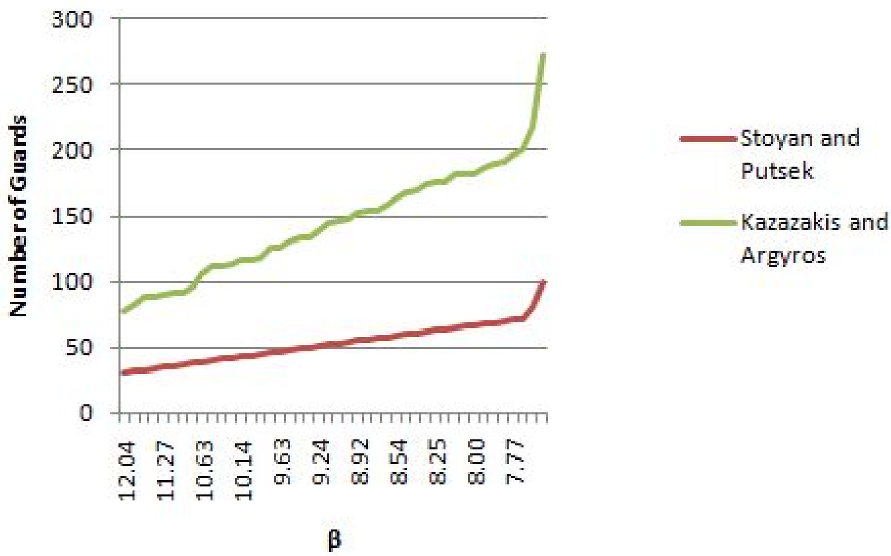


Figure 10 Stoyan and Patsuk's Results [?] vs Kazazakis and Argyros' Results for 100 by 100 square

4.3 Analysis of MOPAD for Full Coverage

4.3.1 Computational Results. The results of the MOPAD methodology for full coverage on the example provided by Kazazakis and Argyros (1081 by 776 square) are shown against the implementation of Kazazakis and Argyros' method in Figure **??** and Appendix **??**. As shown, the MOPAD methodology consistently requires fewer guards for coverage and therefore performs better than Kazazakis and Argyros' method. The MOPAD results for the example provided by Stoyan and Patsuk (100 by 100 square) are shown against Stoyan and Patsuk and the implementation of Kazazakis and Argyros' results in Figure **??** and Appendix **??**. The MOPAD results show consistently more minimal results than Kazazakis and Argyros' method although still overestimate the required number of guards compared to Stoyan and Patsuk. Specifically, while the Kazazakis and Argyros' method requires an average of 170 percent more guards than Stoyan and Patsuk, the MOPAD method requires an average of 68 percent more guards than Stoyan and Patsuk. The

MOPAD methodology appears to be less sensitive to small changes in radius than Stoyan and Patsuk's results, producing a stair-step style pattern.

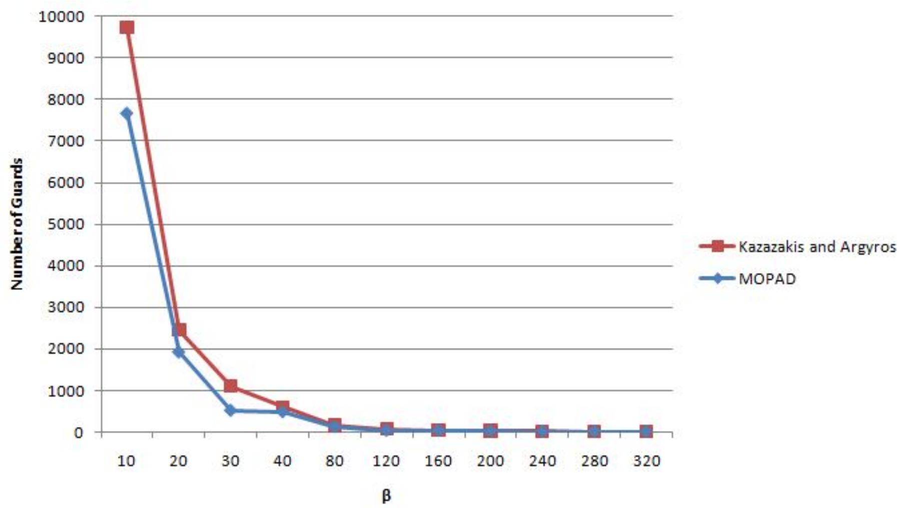


Figure 11 Number of Guards as a Function to Visibility Range: MOPAD vs Kazazakis

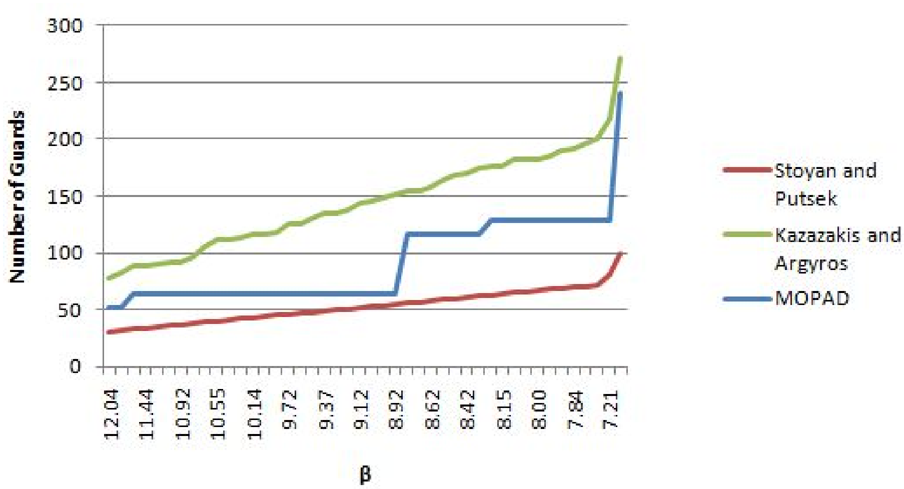


Figure 12 Stoyan's Results [?] vs MOPAD for 100 by 100 square

249 additional combinations of convex polygons and β were tested and the results of the MOPAD methodology and implemented Kazazakis and Argyros methodology on each combination are shown in Appendix ??, ??, ??, and ??. These test cases combined with the cases discussed in Kazazakis and Argyros' and Stoyan and

Patsuk's papers produce 312 separate combinations of P and β. The results of these tests showed that in some instances where 1 or 2 guards are required to cover the polygon, the MOPAD method and Kazazakis and Argyros method produced the same optimal results. In all other instances requiring a larger number of guards (> 2), the MOPAD method produced results ranging between 6.25 percent and 57.89 percent decrease in the required number of guards as compared to Kazazakis and Argyros. Of these results 69.55 percent produced greater than or equal to a 20 percent decrease and 84.29 percent produced a greater than 10 percent decrease.

Stoyan and Patsuk also presented numerical results for a non-convex polygon with holes. Stoyan and Patsuk provide the minimal visibility range required to cover the region with 30 and 40 guards [**?**]. The 30 guard covering with β equal to 16.617665 is shown in Figure **??** [**?**]. The MOPAD method requires an input of a convex polygon with no holes. In order to apply the MOPAD method to the shape provided by Stoyan, the holes were removed from the figure and the convex hull of the polygon was used as an input (Figure **??**). The polygon was shifted into the positive quadrant of the x, y plane. After the results of the MOPAD method were obtained using the convex hull as the input polygon, any guards in parts of the polygon that were not part of the original shape were removed. This process was done manually and required some movement of the guards to insure that any gaps caused by the removal of guards were covered. With the given visibility range, the MOPAD method with manual removal of unnecessary guards to compensate for the non convex input resulted in 33 guards as compared to Stoyan and Patsuk's 30 (Figure **??**), and 44 guards as compared to Stoyan and Patsuk's 40 (Figure **??**). The higher results produced by the MOPAD method may be a result of how unnecessary guards were removed. Stoyan and Patsuk's method allows for non-convex polygons with holes and therefore these features are accounted for with the methodology [**?**]. MOPAD on the other hand does not account for any holes in the polygon. This could result in less efficient placement of some guards around the holes or non-convex edges of P.

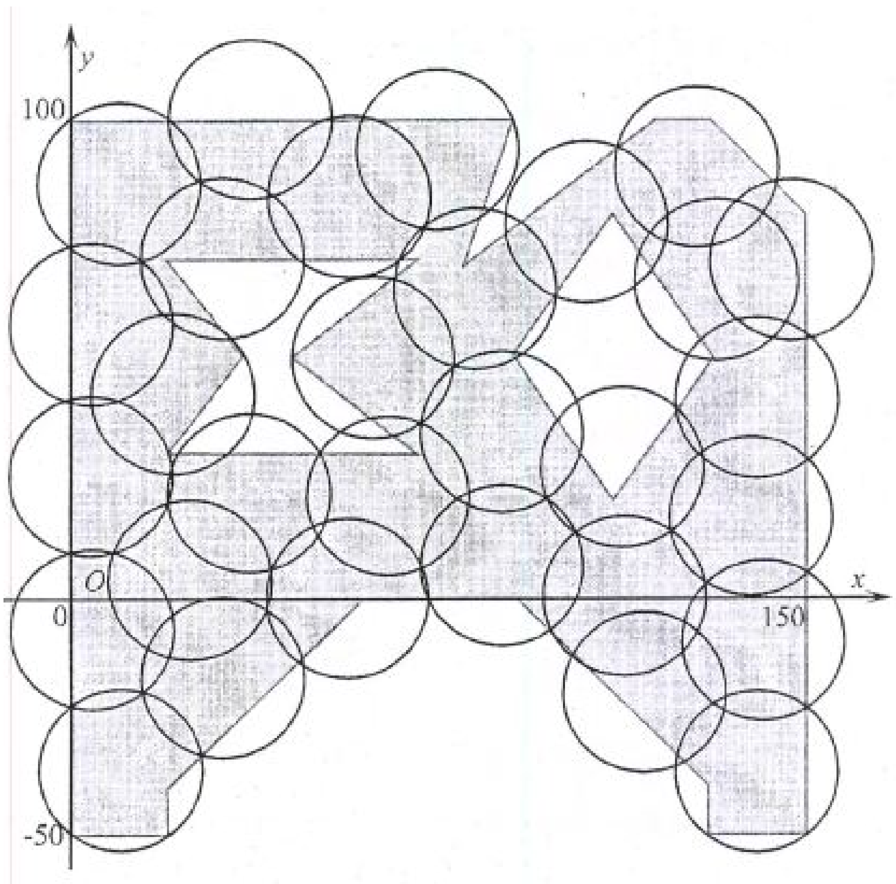


Figure 13 Stoyan's Covering of Region with 30 guards [?]

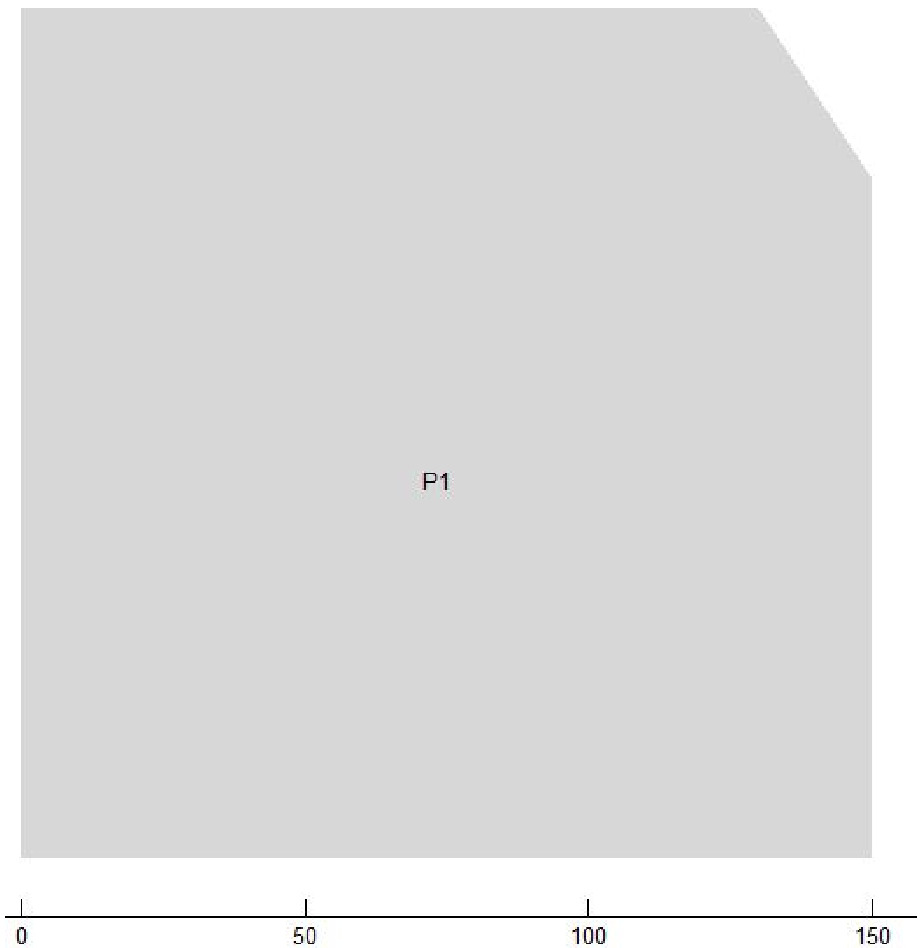


Figure 14 Convex Hull of Stoyan's Example

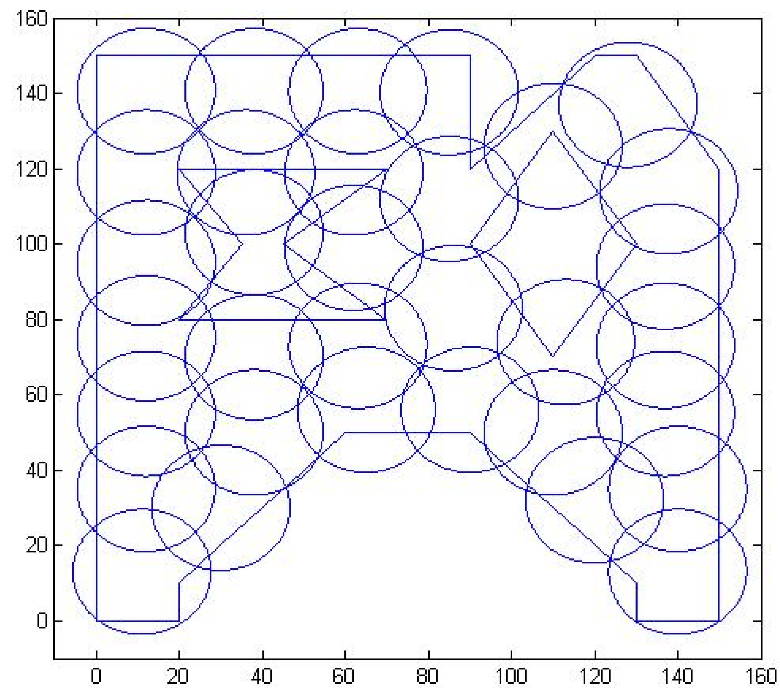


Figure 15 MOPAD Covering of Region with 33 guards

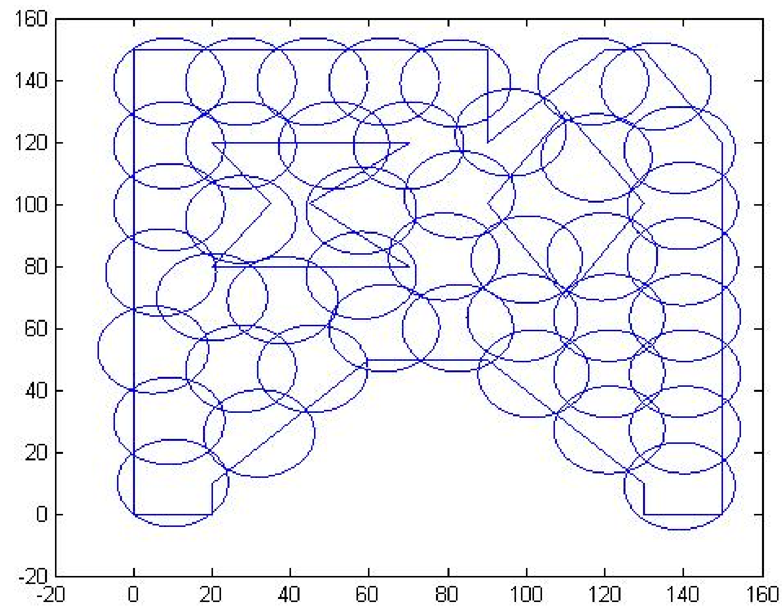


Figure 16 MOPAD Covering of Region with 44 guards

4.3.2 Computational Time Results. When providing analysis of large instances of the problem, such as the NORTHCOM instance, computation time is an important factor in the choice of methodology. The CPU time required to solve each of the 312 instances discussed previously in this section using the implementation of Kazazakis and Argyros' method and the MOPAD method were compared. Figure **??** shows the CPU time required to solve the example presented by Stoyan and Patsuk using Kazazakis and Argyros' method and the MOPAD method. These results are also shown in Appendix **??**, **??**, **??**, **??**, **??**, and **??**. In all instances the MOPAD method was at least as fast as the Kazazakis and Argyros method. Over 96 percent of the test cases produced at least a 50 percent decrease in CPU time required to solve the instance and over 63 percent produced at least 80 percent decrease in time. These results show that the implementation of the MOPAD method described in chapter 3 provides results consistently faster than the implementation of Kazazakis and Argyros method described in chapter 2. These methods were implemented in MATLAB R2009B on a HP Pavilion dv9500 Notebook PC.

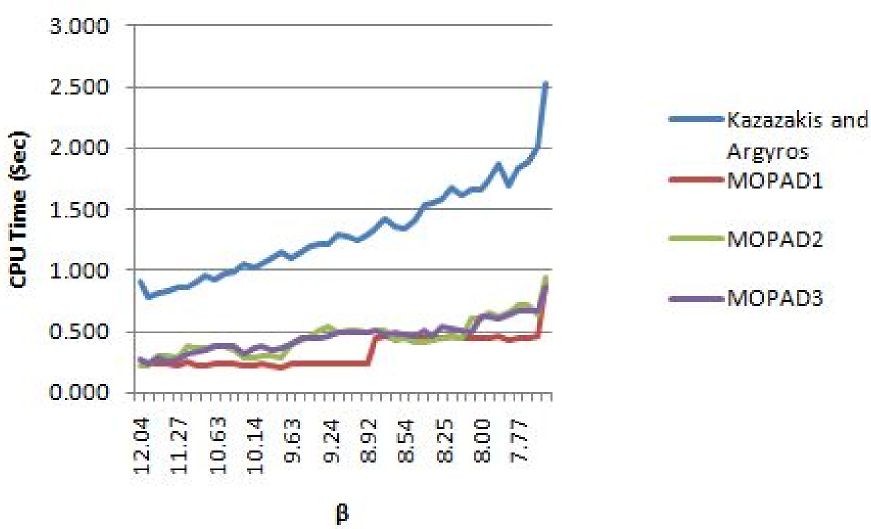


Figure 17 CPU Time required to Solve 100 by 100 Square

4.3.3 Bounding Results. Given a polygon, P, the area of the polygon A_P can be calculated using relatively simple geometry. The area of a circle with radius β is given by $A_{OP} = \pi\beta^2$. A lower bound for the number of guards required to cover the polygon is given by $ceiling(\omega = \frac{A_P}{A_{OP}})$. It is not possible to attain a coverage of P with less than ω guards because the area of the union of the circles around each guard must be at least the area of P. While it is possible for the bound to be optimal in some instances where only one guard is required, this bound will not necessarily be the optimal number. This bound is developed assuming the circles fit together perfectly without covering any area with more than one circle. This is not realistic in a scenario that requires more than one guard because the circles of detection do not fit together perfectly like a puzzle. Consequently ω provides a lower bound that is no worse than the optimal solution but not necessarily optimal.

900 randomly generated polygons were run through the MOPAD methodology using 900 randomly generated β values. These polygons were generated using a MATLAB code developed by Roger Stafford to generate random convex polygons given a number of vertices [?]. The number of vertices range between 3 and 20 and the lower bound ranged between 1 and 19,535. These runs resulted in an average solution of 2.6994ω circles required to cover the polygon. The worst performing scenarios showed a solution of 8ω. These worst case scenarios occurred with a small number of vertices (3) and elongated shapes. (For an example see Figure **??**). In these scenarios the bound is farther from optimal because P has a small area but the elongated shape requires a larger number of circles for full coverage. The specific ω values and number of guards required by MOPAD can be seen in Appendix **??**.

Based on these results a 99 percent one-sided prediction interval can be built as a measure of MOPAD performance. The 900 independent runs had an average result, $\overline{x}$, of 2.726ω and a standard deviation, s, of .693. A one sided upper prediction interval can be built using equation **??** [?]. Based on these results, with 99 percent confidence, the MOPAD method will produce results within 4.34 times ω.

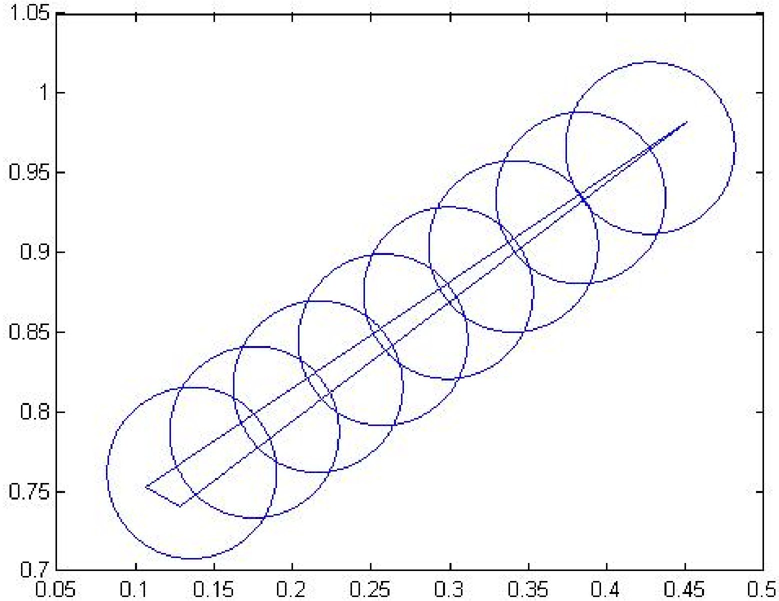
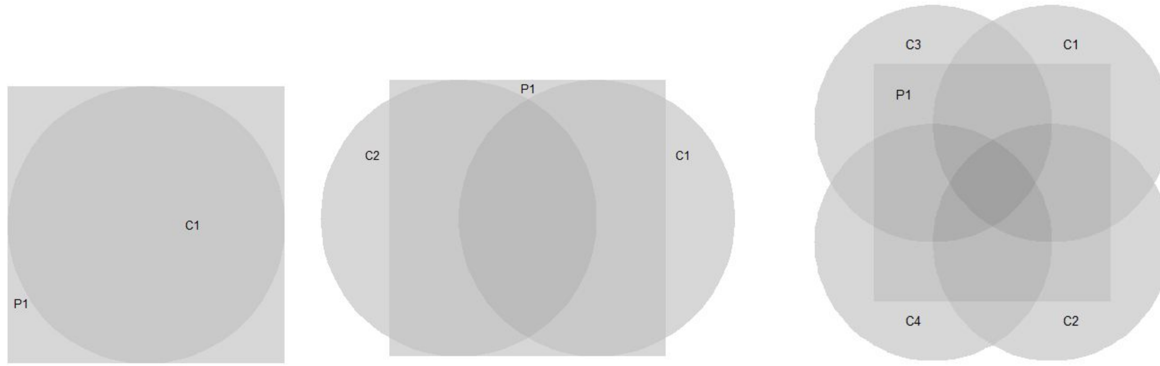


Figure 18 Example of Scenario Resulting in 8ω

$$\overline{x} + t_{\alpha} s \sqrt{1 + 1/n} \tag{19}$$

4.4 Analysis of MOPAD for Partial Coverage

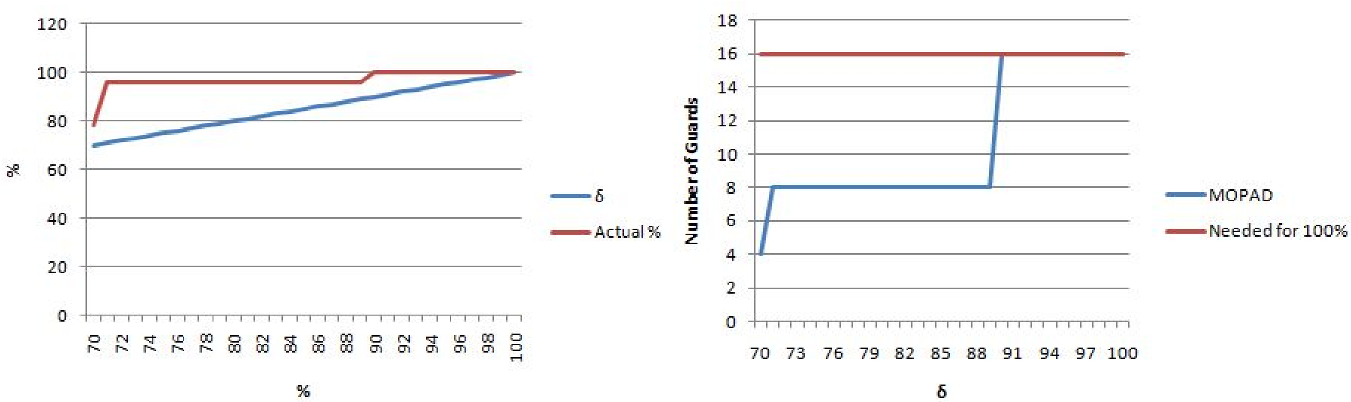
4.4.1 Simple Test Case. $P_1 = [0, 0;\ 2, 0;\ 2, 2;\ 0, 2]$ with $\beta = 1$. One guard should result in $\delta_i = \frac{\pi * 1^2}{2 * 2} * 100 = 78.53$ percent coverage (See Figure **??**(a). Two circles centered at (0.5,1) and (1.5,1) produces coverage of 95.65 percent for each sub-polygon and 95.65 percent for the original polygon (See Figure **??**(b)). The MOPAD method for full coverage produced a result of 4 circles required (Shown in Figure **??** (c)). It can be seen that while the MOPAD for partial coverage produces results with a percentage coverage higher than the requested δ, the number of guards required is reduced as δ is reduced.



(a) 78.53 Percent Coverage (b) 95.65 Percent Coverage (c) 100 Percent Coverage

Figure 19 Partial Coverage Test Scenario 1

Figure ??(a) shows the MOPAD method for partial coverage produced a percent coverage greater than or equal to the requested percent coverage in every case. The number of guards required is reduced by reducing the percent coverage required as shown in Figure ??(b).



(a) Actual Percent Coverage vs δ (b) Guards Required for Partial Coverage vs Full Coverage

Figure 20 Partial Coverage Test Scenario 1

4.4.2 Results for Stoyan Test Case. The MOPAD method for partial coverage was applied to the 100 by 100 square provided in Stoyan and Patsuk's paper. The results are shown in Figure ??. It can be seen in these results that a reduced δ results in a reduced number of guards in most cases. Because the actual percent coverage is not always tight to δ in some instances a δ of 90 percent resulted in the same number of guards as full coverage.

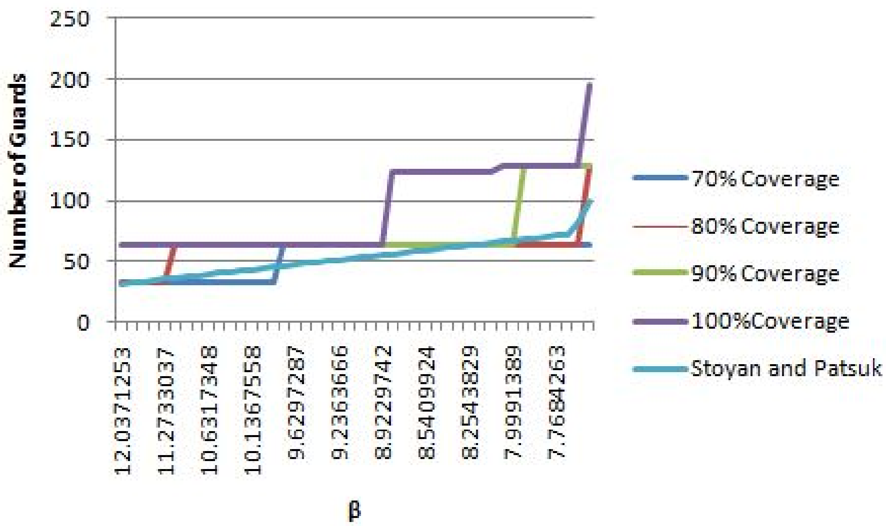


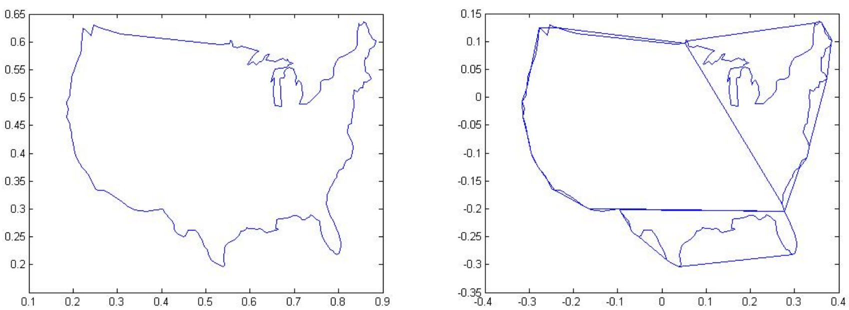
Figure 21 Partial Coverage Results for 100 by 100 Square

4.5 NORTHCOM Instance

Raw data for the locations of the border of CONUS was not readily available. The sponsor suggested that the data be found through the use of GoogleEarth. However, this method could involve significant error. Instead, data was obtained from Steve West [?]. West used the state boundary data set from the U.S. Census Bureau and eliminated duplicate and unnecessary data points [?]. West then converted the latitude and longitude values to an x, y coordinate system and stored the data in an Excel file [?]. The data provided by West is the data used for the NORTHCOM instance of the problem.

Figure ??(a) shows the shape of CONUS used for the purpose of this research. Because this polygon is not convex, the input polygon was split into 3 separate polygons and each sub-polygon was made convex (Figure ??(b)). The MOPAD methodology was then run on each of the sub-polygons with a visibility range of 40

NM. Any guards that provided coverage of an area that is not part of the original CONUS were removed. This produced a requirement for 1,448 MPAR units to provide full coverage (Figure **??**). Additionally, the MOPAD method was run using the same inputs for requirements of 90, 80, and 70 percent coverage (Figure **??**, **??**, and **??** respectively). A reduction of 30 percent coverage resulted in a reduction of over 50 percent for the number of guards required.



(a) CONUS (b) CONUS Divided into Convex Polygons

Figure 22 CONUS Input to MOPAD Method

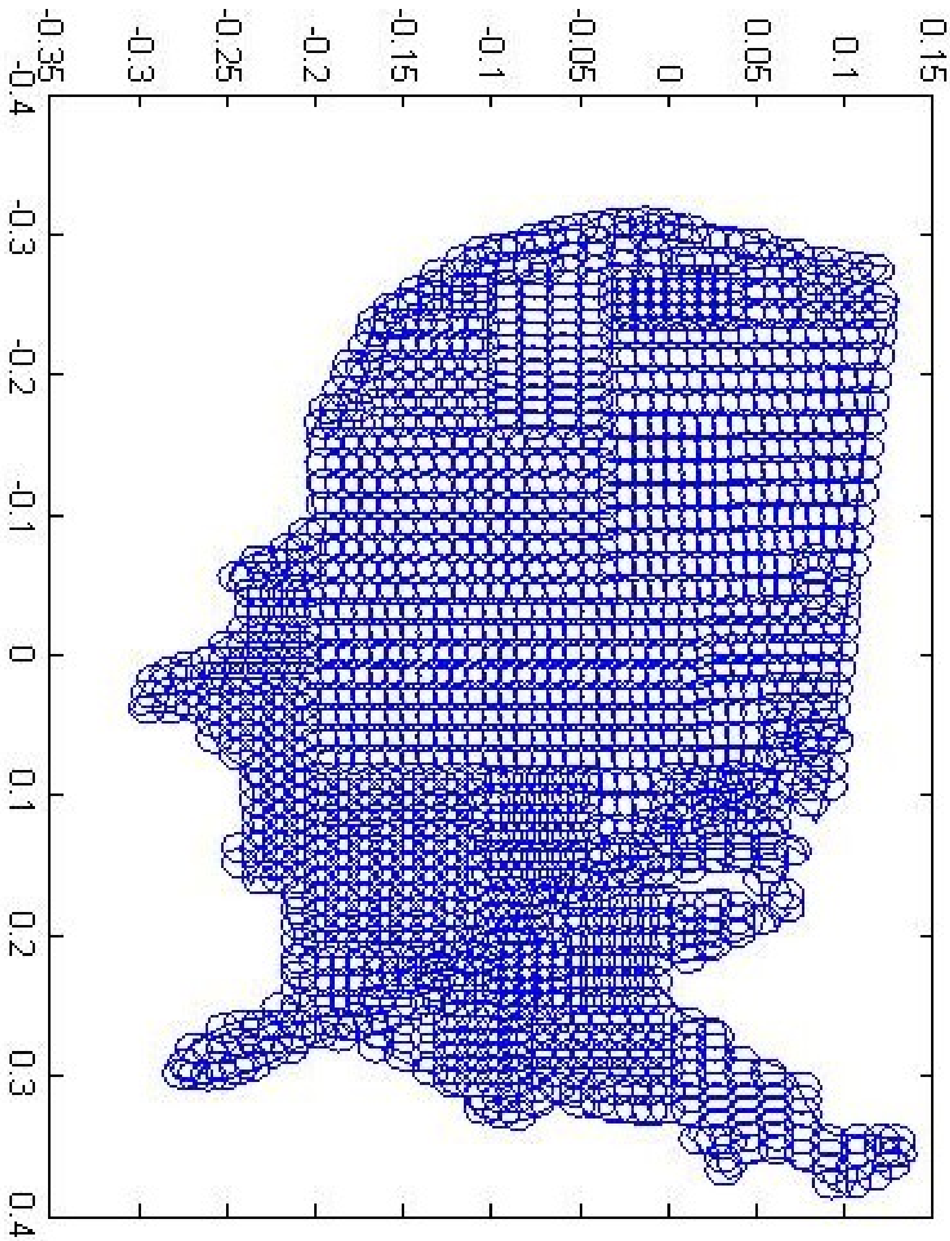


Figure 23 Full Coverage of the United States Using 1448 MPAR

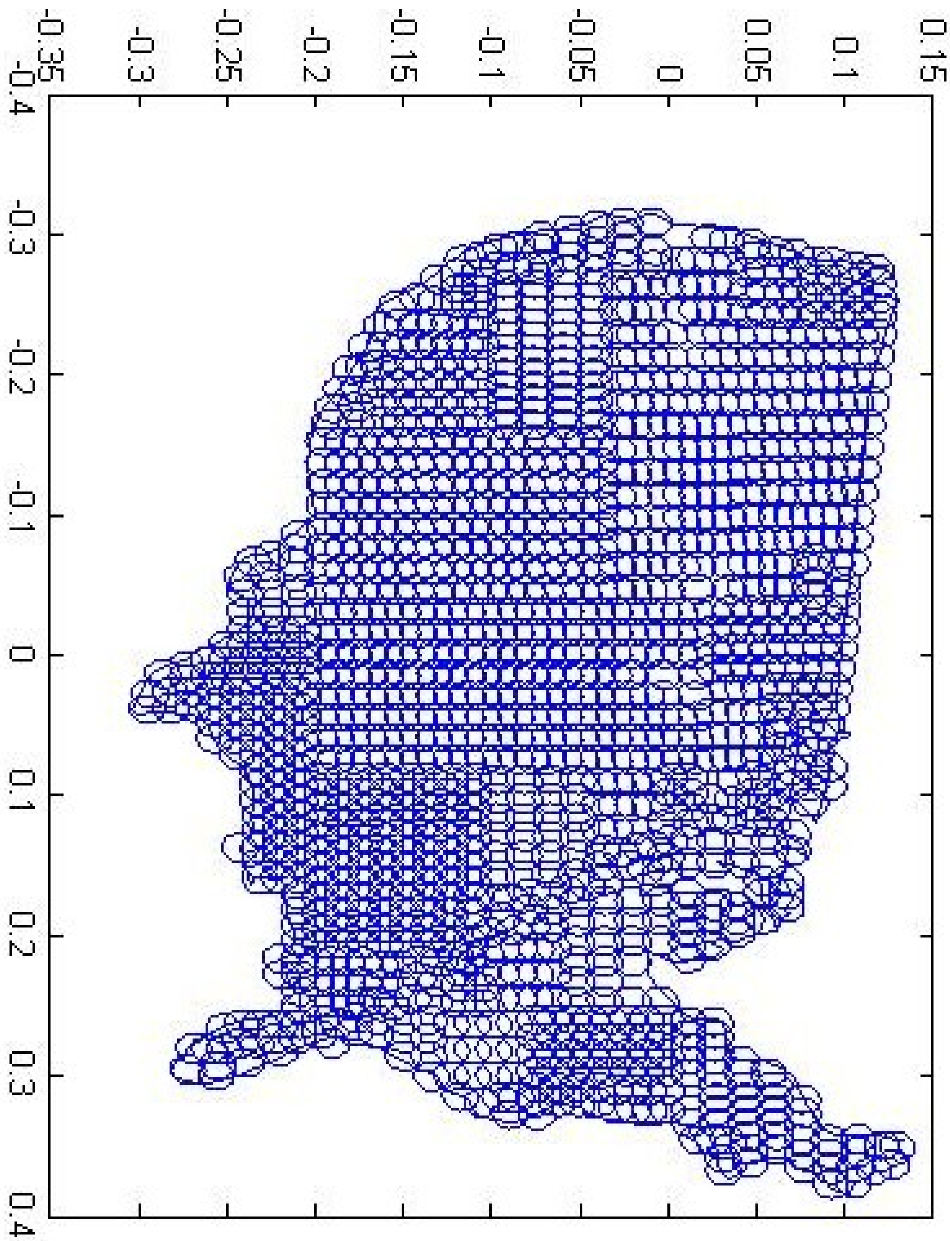


Figure 24 90 Percent Coverage of the United States Using 1282 MPAR

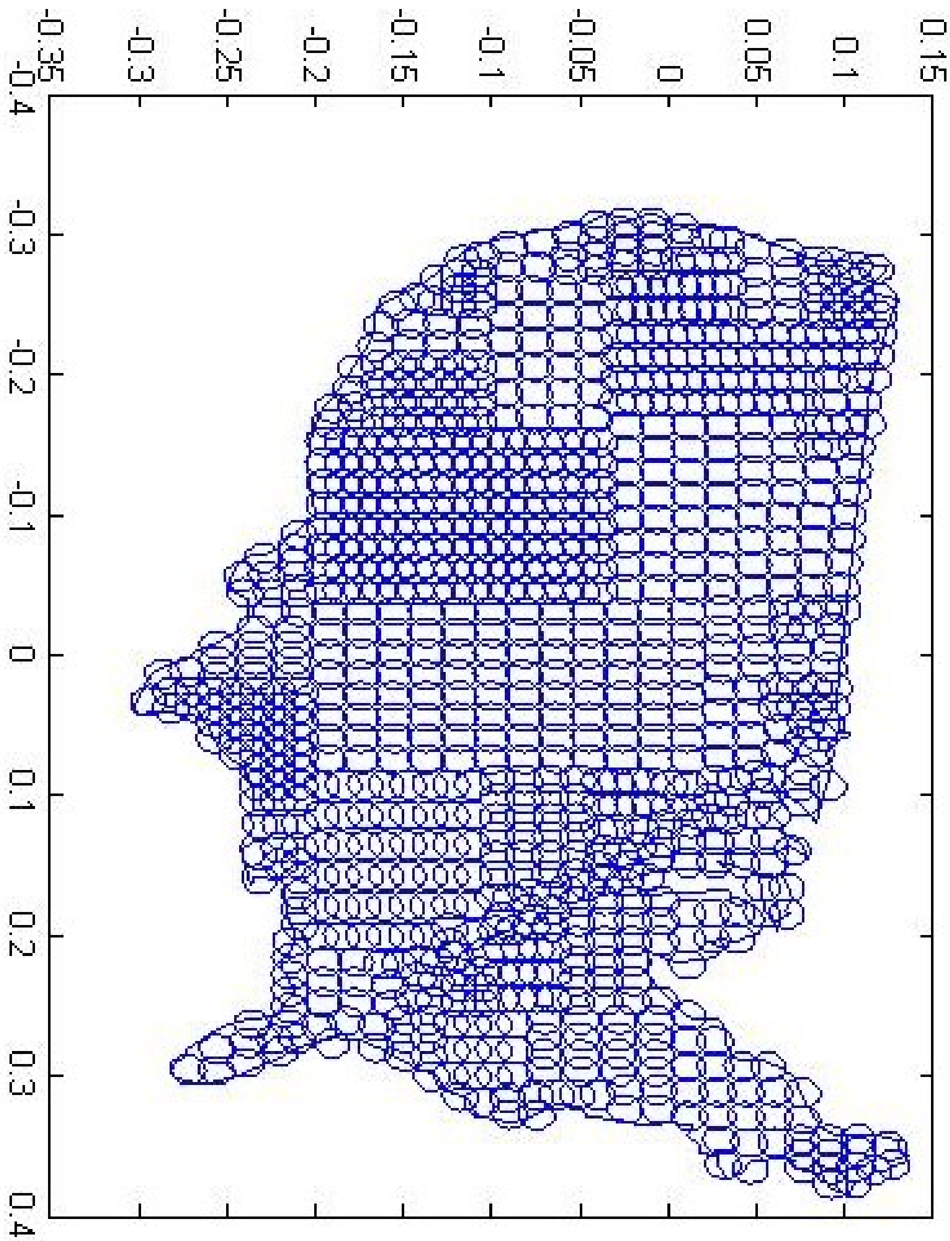


Figure 25 80 Percent Coverage of the United States Using 920 MPAR

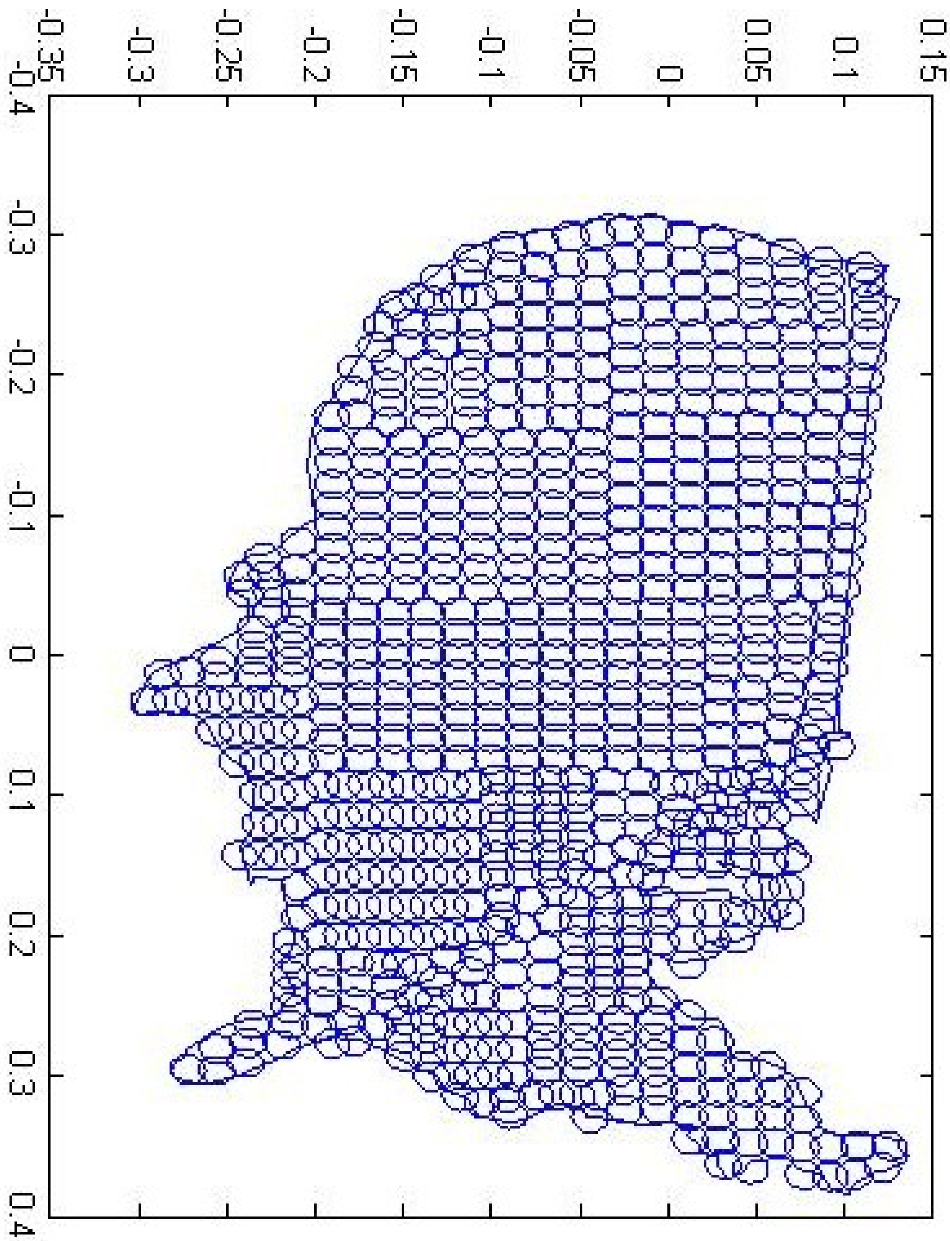
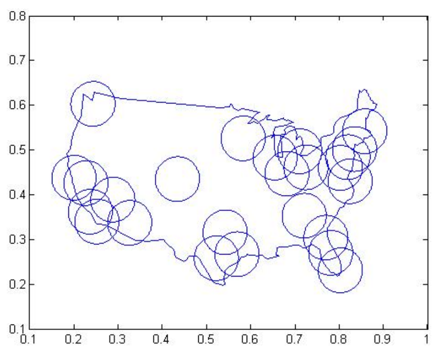
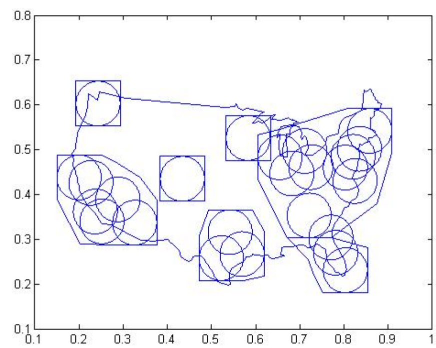


Figure 26   70 Percent Coverage of the United States Using 712 MPAR

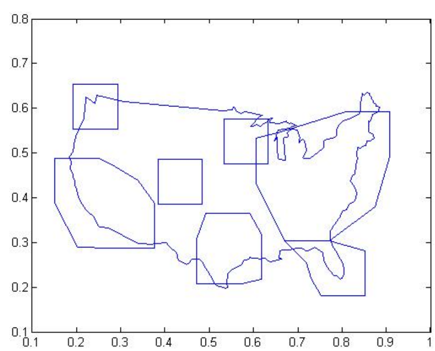
Due to budgetary restraints it is possible that full coverage of CONUS may not be possible. However, there are several cities over which having full coverage is necessary. Figure ??(a) shows these cities and the 200 NM radius around the cities over which coverage is desired. Figure ??(c) shows the polygonal input used as input for the MOPAD method. After results for the input shown in Figure ??(c) were obtained, any guards outside of the area shown in Figure ??(a) were removed. The MOPAD methodology was applied to these areas. These results are shown in Figure ??. Figures ??, ??, ?? show 90, 80, and 70 percent coverings of CONUS respectively with full coverage of the 25 cities of interest.



(a) Key Cities AOS



(b) Key Cities AOS inscribed in MOPAD input



(c) Key Cities MOPAD input

Figure 27 Key Cities

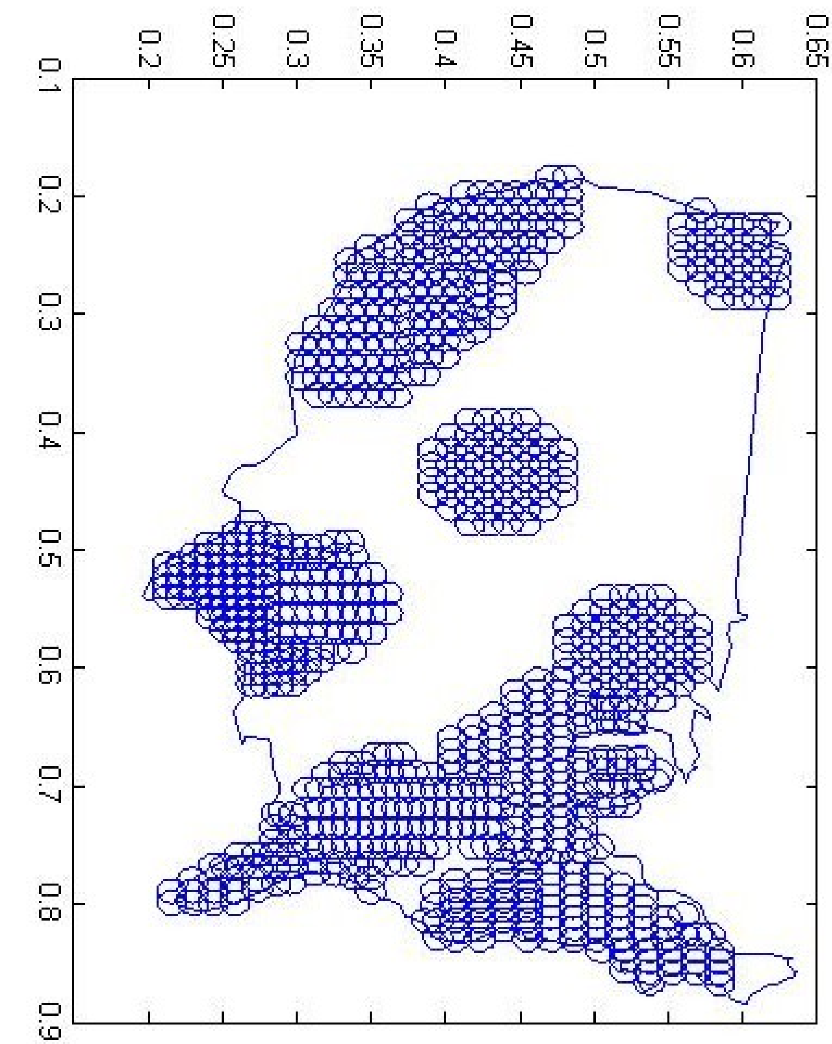


Figure 28   Covering of 25 Cities of Interest Using 710 MPAR

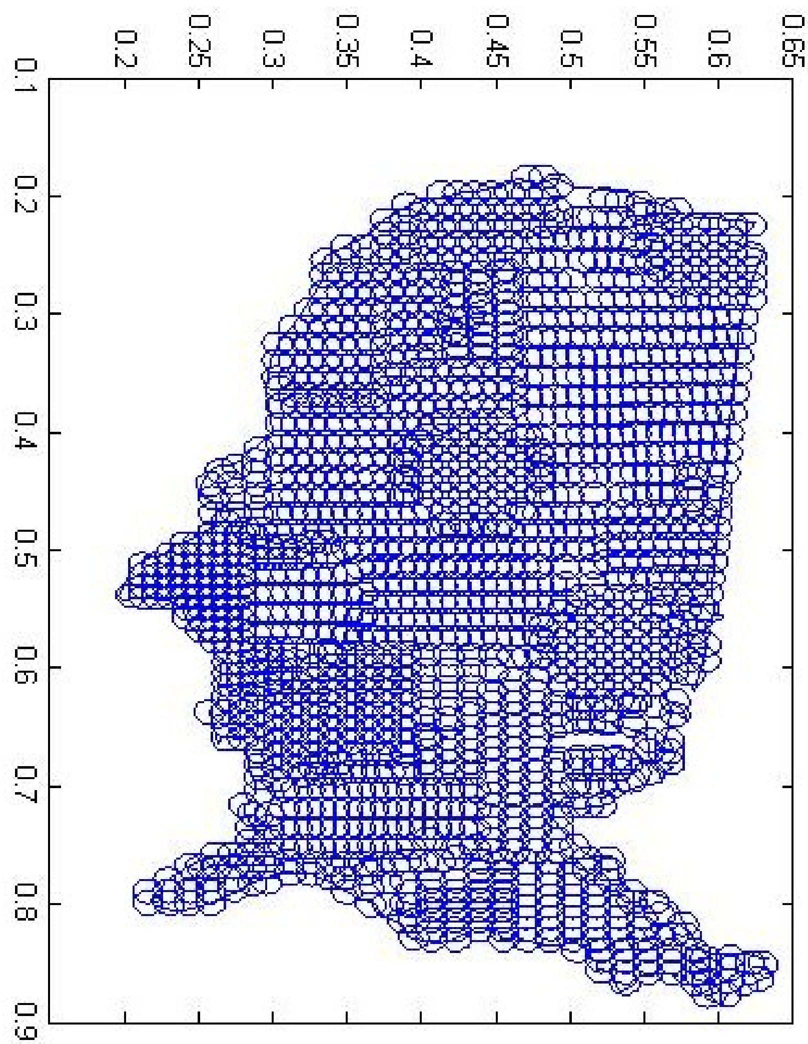


Figure 29 90 Percent Covering of CONUS with Full Coverage of 25 Cities Using 1373 MPAR

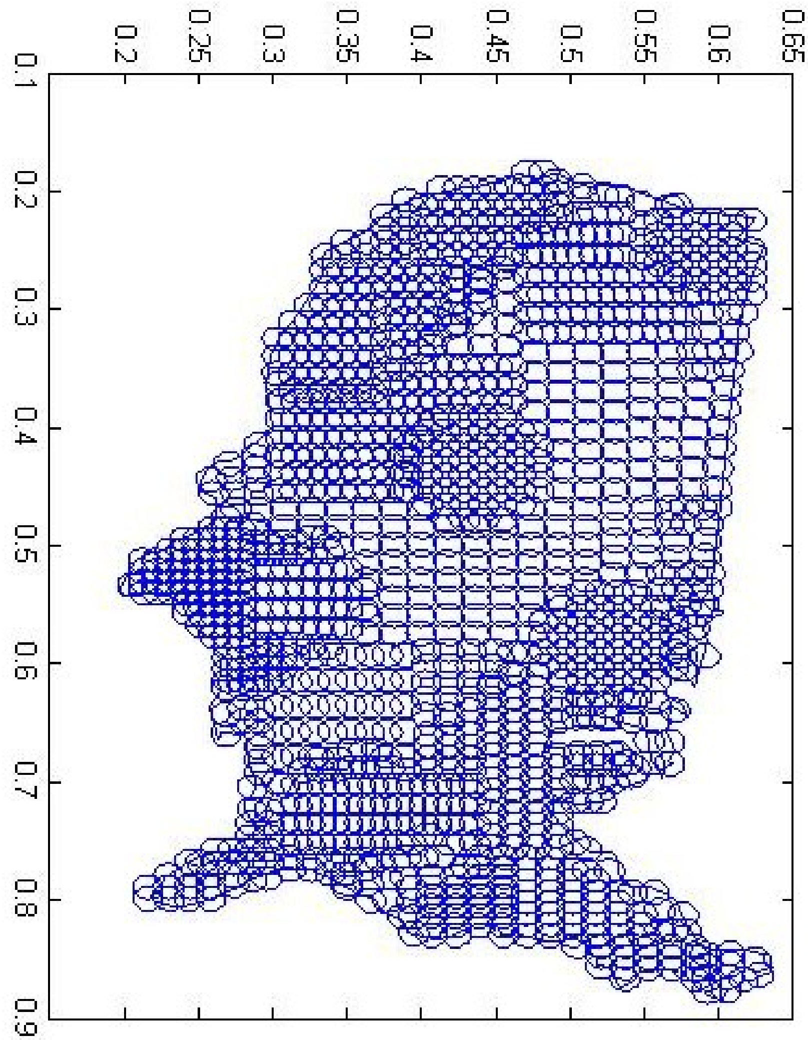


Figure 30 80 Percent Covering of CONUS with Full Coverage of 25 Cities Using 1184 MPAR

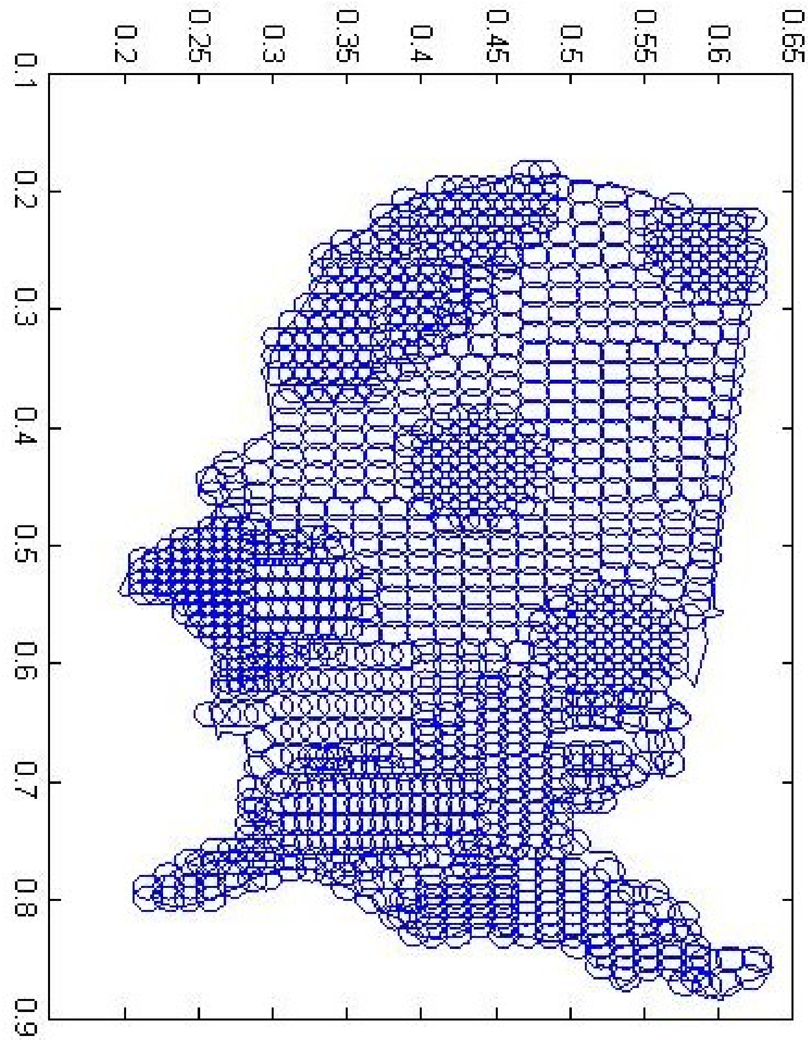


Figure 31 70 Percent Covering of CONUS with Full Coverage of 25 Cities Using 1074 MPAR

4.6 Conclusions

The MOPAD methodology developed in Chapter 3 quickly provides a small set of guards required to cover the interior and exterior of a convex polygon at a percent coverage less than or equal to 100 percent. The MOPAD method consistently provides a smaller number of required guards in a shorter amount of time than the implemented Kazazakis and Argyros method discussed in Chapter 2. The MOPAD method is expected to provide a set of guards within 4.34 times the lower bound in 99 percent of instances.

5. *Conclusions*

5.1 Conclusions

Current literature does not provide a method for quickly determining a small set of limited visbility guards required to cover the interior and exterior of a continuous polygonal region at a level less than or equal to 100 percent. The MOPAD method developed in this thesis quickly provides a good solution to this problem. The MOPAD method consistently requires a smaller number of guards required to cover a region and requires less computational time to solve than the method presented by Kazazakis and Argyros. Additionally, the MOPAD method is expected to require within 4.34 times the lower bound guards 99 percent of the time.

The application of the MOPAD method to NORTHCOM's problem provides a required number of MPAR units needed to cover CONUS and 25 key cities of interest. The solution to this problem, could save the United States significant amounts of money during the replacement process and improve surveillance capabilities of the future NAS system. In order to cover CONUS at a level of 100 percent, 1,448 MPAR units are required. This cost could be prohibitively expensive. A reduction to the number of required MPAR units of 25.8 percent (1,074 MPAR required) can be achieved by providing 100 percent coverage of the 25 cities of interest provided by NORTHCOM and 70 percent coverage of the rest of CONUS.

5.2 Possible Applications

The MOPAD method presented in Chapter 3 is applicable to many additional situations. For example, the method could be applied to determine the location for surveillance resources in forward operating areas where the number and location of these units must be determined quickly and the shape of the region may change often. This method could also be used to determine the location of sensors, lights, mines, microphones, or cameras in areas where trespassing is forbidden. The method is robust enough to provide a good solution to most situations where the area of coverage can be modeled as a polygon and the resource can be modeled as a circle of constant radius.

5.3 Future Work

5.3.1 NORTHCOM Instance. For the purpose of this research, the sponsor requested that terrain features not be considered. Features of the land area such as ownership, buildings, availability, etc were not considered. These are important factors in determining the final location of the MPAR units and should be included for future analysis. MPAR units are considered extremely reliable because the technology is capable of functioning even when the unit is degraded. The sponsor requested that 100 percent reliability of MPAR units be assumed. In practice each MPAR may not be operational 100 percent of the time due to failure or maintenance. Future research should be done to consider the impact of reduced reliability and determine if redundant MPAR units are required.

5.3.2 Methodology. Additional analysis can be conducted to determine a better relationship between λ and κ that will result in a coverage level closer to the requested level. The method could be expanded to handle non-convex polygons, polygons not contained in the positive x, y quadrant, polygons with holes, or shapes other than polygons such as circular regions.

The results of the MOPAD method show redundant coverage in some instances. Further analysis related to how to reduce these redundancies could improve performance of the MOPAD method. Another phase in the method could be implemented to check the current solution for redundancies and remove them. Kazazakis and Argyros discuss the benefits of using the median point for the OP [**?**]. Stoyan and Patsuk mention the importance of OP selection but do not describe the specific method used in their paper [?]. Further analysis of OP selection could be conducted in an effort to improve the MOPAD method and reduce redundant guards.

While the MOPAD method assumes a continuous region, a discrete method may be capable of solving the NORTHCOM and other instances of the problem. Comparing the computation time to solve and numerical results of the MOPAD method to available discrete methods would provide additional measures of the goodness of solution and speed of the method.

Appendix A. Test Cases

A.1 Overview

The following tables and figures show results for test cases used in the analysis presented in Chapter 4. The columns labeled "S and P" represent the results from Stoyan and Patsuk. The columns labeled "K and A" are the results from the authors Matlab implementation of Kazazakis and Argyros method as presented in Chapter 2. MOPAD1, MOPAD2, and MOPAD3 columns are the data for the respective implementation of the MOPAD method and the MOPAD column is the most minimal result of the three implementations. The columns labeled ω show the lower bound value, ω, for the instance as calculated in Chapter 4 and the column MOPADω shows how many times greater than ω the MOPAD results were. All data, except the results of Stoyan and Patsuk, were achieved by implementing the methodologies in Matlab R2009b on a HP Pavilion dv9500 Notebook PC.

A.2 100 by 100 Square Presented in Stoyan and Patsuk Paper

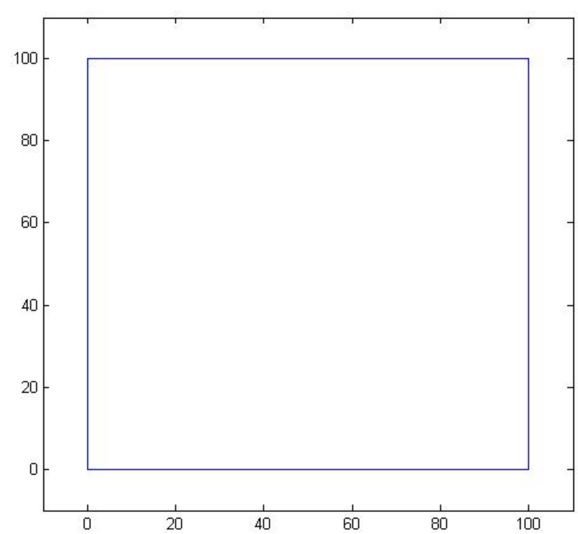


Figure 32 100 by 100 Square Presented by Stoyan and Patsuk

			Required Guards			
β	**S and P [?]**	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**	**MOPAD**
12.04	31	78	64	52	64	52
11.84	32	82	64	52	68	52
11.58	33	88	64	84	68	64
11.44	34	88	64	84	72	64
11.27	35	90	64	84	72	64
11.02	36	92	64	100	84	64
10.92	37	92	64	100	92	64
10.80	38	96	64	100	100	64
10.63	39	106	64	100	100	64
10.55	40	112	64	100	100	64
10.45	41	112	64	100	100	64
10.18	42	114	64	80	96	64
10.14	43	116	64	80	96	64
10.00	44	116	64	80	96	64
9.89	45	118	64	80	96	64
9.72	46	126	64	80	96	64
9.63	47	126	64	112	112	64
9.53	48	130	64	128	120	64
9.37	49	134	64	128	120	64
9.31	50	134	64	144	128	64
9.24	51	138	64	144	128	64
9.12	52	144	64	144	136	64
9.07	53	146	64	144	136	64
9.01	54	148	64	144	144	64
8.92	55	152	64	144	144	64

		Required Guards				
β	**S and P [?]**	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**	**MOPAD**
8.74	56	154	128	144	144	116
8.68	57	154	128	144	144	116
8.62	58	158	128	116	136	116
8.54	59	164	128	116	136	116
8.43	60	168	128	116	136	116
8.42	61	170	128	116	136	116
8.35	62	174	128	116	136	116
8.25	63	176	128	132	144	128
8.15	64	176	128	132	144	128
8.11	65	182	128	132	144	128
8.06	66	182	128	164	144	128
8.00	67	182	128	180	172	128
7.95	68	186	128	180	172	128
7.90	69	190	128	180	172	128
7.84	70	192	128	180	172	128
7.77	71	196	128	196	180	128
7.66	72	200	128	196	180	128
7.21	81	218	128	176	192	128
6.48	100	272	256	256	240	240

Required Guards

Radius	**S and P [?]**	**K and A**	**MOPAD**	ω	**MOPAD**ω
12.04	31	78	52	22	2.36
11.84	32	82	52	23	2.26
11.58	33	88	64	24	2.67
11.44	34	88	64	25	2.56
11.27	35	90	64	26	2.46
11.02	36	92	64	27	2.37
10.92	37	92	64	27	2.37
10.80	38	96	64	28	2.29
10.63	39	106	64	29	2.21
10.55	40	112	64	29	2.21
10.45	41	112	64	30	2.13
10.18	42	114	64	31	2.06
10.14	43	116	64	31	2.06
10.00	44	116	64	32	2.00
9.89	45	118	64	33	1.94
9.72	46	126	64	34	1.88
9.63	47	126	64	35	1.83
9.53	48	130	64	36	1.78
9.37	49	134	64	37	1.73
9.31	50	134	64	37	1.73
9.24	51	138	64	38	1.68
9.12	52	144	64	39	1.64
9.07	53	146	64	39	1.64
9.01	54	148	64	40	1.60
8.92	55	152	64	40	1.60

Required Guards

Radius	**S and P [?]**	**K and A**	**MOPAD**	ω	**MOPAD**ω
8.74	56	154	116	42	2.76
8.68	57	154	116	43	2.70
8.62	58	158	116	43	2.70
8.54	59	164	116	44	2.64
8.43	60	168	116	45	2.58
8.42	61	170	116	45	2.58
8.35	62	174	116	46	2.52
8.25	63	176	128	47	2.72
8.15	64	176	128	48	2.67
8.11	65	182	128	49	2.61
8.06	66	182	128	49	2.61
8.00	67	182	128	50	2.56
7.95	68	186	128	51	2.51
7.90	69	190	128	52	2.46
7.84	70	192	128	52	2.46
7.77	71	196	128	53	2.42
7.66	72	200	128	55	2.33
7.21	81	218	128	62	2.06
6.48	100	272	240	76	3.16

Computational Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
12.04	0.905	0.265	0.218	0.265
11.84	0.780	0.234	0.218	0.234
11.58	0.811	0.234	0.296	0.281
11.44	0.827	0.234	0.296	0.250
11.27	0.858	0.218	0.281	0.265
11.02	0.858	0.250	0.374	0.312
10.92	0.905	0.218	0.359	0.328
10.80	0.952	0.218	0.359	0.343
10.63	0.920	0.234	0.374	0.374
10.55	0.967	0.234	0.374	0.374
10.45	0.983	0.234	0.343	0.374
10.18	1.045	0.218	0.281	0.312
10.14	1.014	0.218	0.281	0.359
10.00	1.045	0.234	0.296	0.374
9.89	1.108	0.218	0.296	0.343
9.72	1.154	0.203	0.281	0.359
9.63	1.108	0.234	0.390	0.390
9.53	1.154	0.234	0.421	0.437
9.37	1.201	0.234	0.468	0.437
9.31	1.217	0.234	0.515	0.437
9.24	1.217	0.234	0.546	0.468
9.12	1.295	0.234	0.499	0.499
9.07	1.279	0.234	0.515	0.499
9.01	1.248	0.234	0.515	0.499
8.92	1.295	0.234	0.499	0.499

Computational Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
8.74	1.342	0.437	0.515	0.515
8.68	1.420	0.468	0.515	0.484
8.62	1.357	0.437	0.421	0.499
8.54	1.342	0.468	0.437	0.484
8.43	1.404	0.437	0.406	0.468
8.42	1.529	0.437	0.406	0.515
8.35	1.544	0.452	0.421	0.468
8.25	1.576	0.452	0.452	0.546
8.15	1.669	0.468	0.468	0.530
8.11	1.607	0.437	0.452	0.515
8.06	1.654	0.437	0.608	0.499
8.00	1.654	0.452	0.608	0.624
7.95	1.747	0.452	0.655	0.624
7.90	1.872	0.468	0.624	0.608
7.84	1.685	0.421	0.655	0.640
7.77	1.841	0.452	0.718	0.671
7.66	1.888	0.452	0.718	0.671
7.21	2.012	0.468	0.640	0.671
6.48	2.527	0.889	0.936	0.858

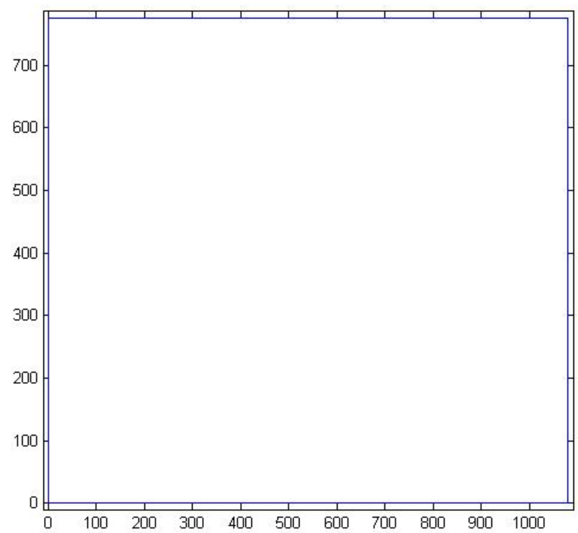


Figure 33 1081 by 776 Square Presented by Kazazakis and Argyros

A.3 1081 by 776 Square Presented by Kazazakis and Argyros

			Required Guards				
β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**	**MOPAD**	ω	**MOPADω**
10	9732	8192	7680	8704	7680	2671	2.88
20	2452	2048	1920	2176	1920	668	2.87
30	1104	512	1120	992	512	297	1.72
40	600	512	480	544	480	167	2.87
80	158	128	120	136	120	42	2.86
120	70	32	70	62	32	19	1.68
160	38	32	30	34	30	11	2.73
200	26	16	24	24	16	7	2.29
240	16	8	10	14	8	5	1.60
280	12	8	8	8	8	4	2.00
320	10	8	8	6	6	3	2.00

Computational Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
10	121.182	46.161	42.651	51.371
20	24.196	8.143	7.722	9.048
30	10.312	1.841	4.290	3.822
40	5.881	1.841	1.778	2.044
80	1.373	0.468	0.437	0.484
120	0.702	0.109	0.281	0.250
160	0.359	0.109	0.094	0.125
200	0.234	0.047	0.109	0.094
240	0.203	0.047	0.062	0.062
280	0.109	0.047	0.031	0.047
320	0.125	0.047	0.031	0.047

A.4 P=[100,0;400,0;500,100;600,200;200,300;100,300;0,200;0,100]

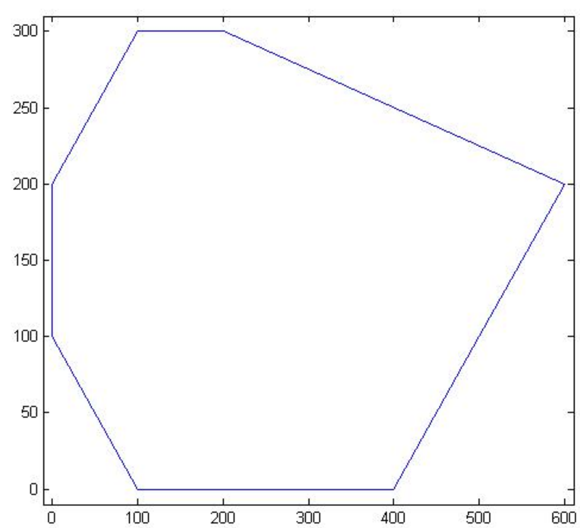


Figure 34 P=[100,0;400,0;500,100;600,200;200,300;100,300;0,200;0,100]

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
1		113824	109037	111112	109037	44564	2.44
2		28555	26573	27038	26573	11141	2.38
3		12013	11763	12447	11763	4952	2.37
4	10355	7189	7537	8362	7189	2786	2.58
5	6580	3919	5115	5286	3919	1783	2.19
6	4549	3048	3510	3746	3048	1238	2.46
7	3402	1976	2872	2718	1976	910	2.17
8	2582	1824	1884	2087	1824	697	2.61
9	2052	1404	1513	1570	1404	551	2.54
10	1668	995	1304	1327	995	446	2.23
11	1362	952	1089	1109	952	369	2.57
12	1149	774	926	951	774	310	2.49
13	976	533	762	796	533	264	2.01
14	831	506	722	686	506	228	2.21
15	739	494	491	581	491	199	2.46
16	646	468	475	530	468	175	2.67
17	568	437	454	451	437	155	2.81
18	510	363	360	398	360	138	2.60
19	458	268	332	362	268	124	2.16
20	408	254	299	320	254	112	2.26
21	385	246	263	304	246	102	2.41
22	333	244	286	286	244	93	2.62
23	302	214	273	263	214	85	2.51
24	289	205	213	231	205	78	2.62
25	263	157	183	215	157	72	2.18

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
26	251	142	197	208	142	66	2.15
27	230	133	195	190	133	62	2.14
28	212	131	184	179	131	57	2.29
29	194	129	139	161	129	53	2.43
30	186	127	129	147	127	50	2.54
31	170	122	127	139	122	47	2.59
32	167	122	121	131	121	44	2.75
33	148	120	118	121	118	41	2.87
34	140	116	115	110	110	39	2.82
35	126	101	97	104	97	37	2.62
36	130	97	84	102	84	35	2.4
37	124	81	94	97	81	33	2.45
38	111	73	92	95	73	31	2.35
39	101	68	80	92	68	30	2.26
40	104	67	78	87	67	28	2.39
41	97	66	72	77	66	27	2.44
42	92	65	73	78	65	26	2.5
43	91	65	69	76	65	25	2.6
44	79	65	70	73	65	24	2.70
45	78	64	67	71	64	23	2.78
46	77	57	60	69	57	22	2.59
47	70	56	59	65	56	21	2.66
48	69	55	55	62	55	20	2.75
49	69	45	53	62	45	19	2.36
50	64	43	51	57	43	18	2.38

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
51	66	39	52	55	39	18	2.16
52	58	38	52	52	38	17	2.23
53	57	35	53	51	35	16	2.18
54	58	34	48	47	34	16	2.12
55	51	34	44	45	34	15	2.26
56	50	34	42	43	34	15	2.26
57	48	34	41	43	34	14	2.42
58	46	34	36	43	34	14	2.42
59	45	34	35	43	34	13	2.61
60	43	33	35	42	33	13	2.53
61	47	32	33	39	32	12	2.66
62	43	32	36	37	32	12	2.66
63	46	32	31	33	31	12	2.58
64	40	32	31	33	31	11	2.81
65	40	32	33	33	32	11	2.90
66	39	32	31	32	31	11	2.81
67	36	32	32	32	32	10	3.2
68	32	32	32	30	30	10	3
69	31	31	29	30	29	10	2.9
70	32	28	27	30	27	10	2.7
71	32	27	25	28	25	9	2.77
72	30	27	25	28	25	9	2.77
73	26	24	24	26	24	9	2.66
74	29	23	21	25	21	9	2.33
75	29	22	22	25	22	8	2.75

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
76	26	22	23	25	22	8	2.75
77	28	21	22	24	21	8	2.62
78	24	21	22	23	21	8	2.62
79	25	21	22	22	21	8	2.62
80	26	21	21	22	21	7	3
81	25	21	23	22	21	7	3
82	24	19	20	19	19	7	2.71
83	23	18	20	19	18	7	2.57
84	23	18	20	18	18	7	2.57
85	25	18	19	17	17	7	2.42
86	20	18	17	17	17	7	2.42
87	22	18	16	17	16	6	2.66
88	22	17	16	17	16	6	2.66
89	21	16	16	17	16	6	2.66
90	19	16	16	17	16	6	2.66
91	18	15	16	17	15	6	2.5
92	19	15	16	17	15	6	2.5
93	22	15	15	17	15	6	2.5
94	19	15	15	16	15	6	2.5
95	17	15	15	16	15	5	3
96	16	15	15	16	15	5	3
97	16	14	15	16	14	5	2.8
98	15	14	15	15	14	5	2.8
99	15	13	15	15	13	5	2.6
100	13	13	15	13	13	5	2.6

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
101	13	13	14	13	13	5	2.6
102	14	12	12	13	12	5	2.4
103	16	12	12	13	12	5	2.4
104	16	12	12	13	12	5	2.4
105	12	12	12	13	12	5	2.4
106	16	11	12	12	11	4	2.75
107	16	11	12	12	11	4	2.75
108	13	10	12	12	10	4	2.5
109	10	9	12	12	9	4	2.25
110	13	9	12	12	9	4	2.25
111	13	9	11	12	9	4	2.25
112	12	9	11	12	9	4	2.25
113	11	9	11	12	9	4	2.25
114	12	9	11	11	9	4	2.25
115	11	9	11	11	9	4	2.25
116	13	9	11	11	9	4	2.25
117	10	9	11	11	9	4	2.25
118	14	8	11	11	8	4	2
119	12	8	10	11	8	4	2
120	10	8	10	11	8	4	2
121	12	8	10	11	8	4	2
122	9	8	10	11	8	3	2.66
123	8	8	9	11	8	3	2.66
124	12	8	9	11	8	3	2.66
125	9	8	9	11	8	3	2.66

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
126	11	8	9	9	8	3	2.66
127	12	8	9	9	8	3	2.66
128	8	8	9	9	8	3	2.66
129	9	8	9	9	8	3	2.66
130	12	8	9	9	8	3	2.66
131	12	8	9	8	8	3	2.66
132	8	8	9	8	8	3	2.66
133	9	8	9	8	8	3	2.66
134	8	8	9	8	8	3	2.66
135	8	8	9	8	8	3	2.66
136	9	8	9	8	8	3	2.66
137	9	8	9	8	8	3	2.66
138	8	8	9	7	7	3	2.33
139	8	8	9	6	6	3	2
140	9	8	9	6	6	3	2
141	8	8	9	6	6	3	2
142	8	8	9	6	6	3	2
143	8	8	9	6	6	3	2
144	8	8	9	6	6	3	2
145	8	8	9	6	6	3	2
146	9	8	9	6	6	3	2
147	7	8	8	6	6	3	2
148	7	8	8	6	6	3	2
149	9	8	7	6	6	3	2
150	8	8	6	6	6	2	3

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
151	7	8	6	6	6	2	3
152	7	8	5	6	5	2	2.5
153	7	8	5	6	5	2	2.5
154	7	8	5	6	5	2	2.5
155	7	8	5	6	5	2	2.5
156	6	8	5	6	5	2	2.5
157	6	8	5	6	5	2	2.5
158	6	7	5	6	5	2	2.5
159	6	6	5	6	5	2	2.5
160	7	6	5	6	5	2	2.5
161	6	6	5	6	5	2	2.5
162	6	6	6	6	6	2	3
163	5	5	5	6	5	2	2.5
164	6	5	5	5	5	2	2.5
165	8	5	5	5	5	2	2.5
166	5	4	4	5	4	2	2
167	6	4	4	5	4	2	2
168	7	4	4	5	4	2	2
169	5	4	4	5	4	2	2
170	4	4	4	5	4	2	2
171	5	4	4	5	4	2	2
172	7	4	4	5	4	2	2
173	6	4	4	5	4	2	2
174	4	4	4	5	4	2	2
175	4	4	4	5	4	2	2

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
176	4	4	4	5	4	2	2
177	4	4	4	5	4	2	2
178	4	4	4	5	4	2	2
179	4	4	4	5	4	2	2
180	4	4	4	5	4	2	2
181	4	4	4	5	4	2	2
182	4	4	4	4	4	2	2
183	6	4	4	3	3	2	1.5
184	5	4	4	3	3	2	1.5
185	4	4	4	3	3	2	1.5
186	5	4	4	3	3	2	1.5
187	4	4	4	3	3	2	1.5
188	4	4	4	3	3	2	1.5
189	4	4	4	3	3	2	1.5
190	5	4	4	3	3	2	1.5
191	4	3	3	3	3	2	1.5
192	4	3	3	3	3	2	1.5
193	4	3	3	3	3	2	1.5
194	4	3	3	3	3	2	1.5
195	4	3	3	3	3	2	1.5
196	4	3	3	3	3	2	1.5
197	4	3	3	3	3	2	1.5
198	4	3	3	3	3	2	1.5
199	4	3	3	3	3	2	1.5
200	4	3	3	3	3	2	1.5

Computation Time (Sec)

β	K and A	MOPAD1	MOPAD2	MOPAD3
4	116.673	39.109	42.276	48.906
5	67.720	17.862	25.210	26.364
6	44.398	13.182	15.772	17.223
7	32.448	8.003	12.480	11.747
8	24.149	7.238	7.535	8.658
9	18.892	5.476	6.037	6.334
10	15.179	3.744	5.101	5.257
11	12.308	3.572	4.290	4.306
12	10.358	2.886	3.526	3.666
13	8.642	1.919	2.855	3.011
14	7.441	1.856	2.730	2.605
15	6.536	1.763	1.778	2.168
16	5.678	1.700	1.747	1.997
17	5.023	1.607	1.669	1.669
18	4.508	1.310	1.326	1.451
19	4.087	0.952	1.232	1.357
20	3.619	0.920	1.076	1.170
21	3.432	0.905	0.967	1.123
22	2.995	0.858	1.045	1.061
23	2.683	0.780	0.967	0.952
24	2.558	0.718	0.749	0.858
25	2.356	0.577	0.686	0.796

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
26	2.246	0.484	0.702	0.780
27	1.997	0.499	0.702	0.686
28	1.872	0.484	0.655	0.655
29	1.716	0.452	0.499	0.593
30	1.622	0.452	0.484	0.515
31	1.544	0.421	0.437	0.515
32	1.482	0.468	0.437	0.484
33	1.326	0.437	0.437	0.421
34	1.248	0.406	0.421	0.406
35	1.139	0.374	0.374	0.374
36	1.139	0.328	0.281	0.374
37	1.092	0.312	0.359	0.374
38	0.998	0.234	0.343	0.343
39	0.889	0.250	0.312	0.359
40	0.936	0.234	0.281	0.359
41	0.858	0.250	0.281	0.281
42	0.796	0.281	0.265	0.281
43	0.811	0.218	0.250	0.296
44	0.702	0.250	0.250	0.265
45	0.718	0.203	0.265	0.250
46	0.702	0.218	0.218	0.265
47	0.655	0.218	0.218	0.265
48	0.608	0.218	0.187	0.218
49	0.608	0.156	0.218	0.234
50	0.577	0.156	0.203	0.203

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
51	0.577	0.140	0.203	0.218
52	0.515	0.125	0.187	0.203
53	0.530	0.140	0.187	0.187
54	0.562	0.140	0.172	0.172
55	0.484	0.109	0.187	0.156
56	0.468	0.140	0.140	0.156
57	0.452	0.140	0.172	0.187
58	0.437	0.125	0.125	0.156
59	0.406	0.125	0.140	0.156
60	0.421	0.109	0.156	0.156
61	0.421	0.125	0.109	0.140
62	0.374	0.140	0.156	0.140
63	0.421	0.109	0.109	0.109
64	0.359	0.109	0.109	0.140
65	0.374	0.109	0.109	0.125
66	0.390	0.109	0.140	0.094
67	0.328	0.140	0.109	0.140
68	0.312	0.109	0.109	0.125
69	0.296	0.125	0.109	0.109
70	0.281	0.125	0.094	0.109
71	0.296	0.109	0.109	0.125
72	0.265	0.125	0.109	0.094
73	0.265	0.078	0.109	0.109
74	0.265	0.094	0.062	0.109
75	0.265	0.109	0.094	0.109

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
76	0.265	0.078	0.109	0.109
77	0.250	0.094	0.078	0.109
78	0.250	0.094	0.078	0.078
79	0.234	0.094	0.078	0.094
80	0.234	0.094	0.094	0.094
81	0.250	0.078	0.094	0.109
82	0.234	0.062	0.094	0.047
83	0.234	0.094	0.078	0.078
84	0.234	0.062	0.094	0.062
85	0.234	0.047	0.062	0.078
86	0.203	0.062	0.078	0.078
87	0.218	0.062	0.078	0.078
88	0.234	0.062	0.078	0.078
89	0.187	0.047	0.047	0.078
90	0.203	0.078	0.047	0.078
91	0.172	0.078	0.078	0.062
92	0.203	0.062	0.078	0.062
93	0.203	0.078	0.062	0.062
94	0.172	0.078	0.078	0.078
95	0.156	0.047	0.078	0.047
96	0.172	0.078	0.047	0.078
97	0.140	0.047	0.047	0.078
98	0.172	0.047	0.078	0.078
99	0.140	0.062	0.062	0.062
100	0.125	0.062	0.047	0.047

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
101	0.125	0.062	0.078	0.047
102	0.125	0.062	0.047	0.047
103	0.172	0.062	0.047	0.062
104	0.172	0.047	0.031	0.047
105	0.125	0.047	0.062	0.031
106	0.172	0.031	0.062	0.062
107	0.172	0.062	0.062	0.078
108	0.140	0.031	0.062	0.047
109	0.109	0.047	0.062	0.031
110	0.140	0.047	0.047	0.062
111	0.140	0.047	0.031	0.062
112	0.140	0.047	0.062	0.062
113	0.094	0.062	0.062	0.062
114	0.109	0.031	0.031	0.062
115	0.125	0.016	0.062	0.031
116	0.125	0.031	0.047	0.062
117	0.109	0.047	0.031	0.062
118	0.156	0.047	0.062	0.031
119	0.109	0.047	0.062	0.031
120	0.125	0.047	0.016	0.062
121	0.140	0.047	0.062	0.047
122	0.094	0.047	0.031	0.031
123	0.078	0.047	0.031	0.062
124	0.109	0.031	0.062	0.031
125	0.109	0.031	0.031	0.062

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
126	0.109	0.031	0.031	0.047
127	0.109	0.047	0.031	0.062
128	0.078	0.047	0.047	0.031
129	0.109	0.047	0.047	0.031
130	0.125	0.047	0.031	0.031
131	0.109	0.047	0.031	0.047
132	0.094	0.031	0.031	0.047
133	0.109	0.031	0.047	0.047
134	0.078	0.016	0.047	0.047
135	0.094	0.047	0.062	0.047
136	0.109	0.047	0.031	0.047
137	0.109	0.047	0.031	0.047
138	0.078	0.047	0.031	0.047
139	0.078	0.047	0.047	0.047
140	0.078	0.031	0.047	0.016
141	0.094	0.031	0.047	0.047
142	0.109	0.016	0.062	0.047
143	0.078	0.047	0.031	0.016
144	0.078	0.031	0.031	0.047
145	0.078	0.047	0.031	0.016
146	0.094	0.047	0.047	0.047
147	0.094	0.047	0.047	0.031
148	0.062	0.047	0.047	0.016
149	0.109	0.047	0.047	0.047
150	0.094	0.047	0.016	0.016

Computation Time (Sec)

β	K and A	MOPAD1	MOPAD2	MOPAD3
151	0.094	0.031	0.016	0.047
152	0.062	0.031	0.047	0.016
153	0.094	0.031	0.016	0.031
154	0.062	0.031	0.031	0.016
155	0.094	0.031	0.016	0.047
156	0.062	0.031	0.016	0.016
157	0.062	0.031	0.031	0.047
158	0.109	0.016	0.016	0.016
159	0.078	0.047	0.016	0.047
160	0.062	0.016	0.031	0.016
161	0.078	0.047	0.031	0.047
162	0.078	0.016	0.047	0.016
163	0.047	0.031	0.016	0.047
164	0.078	0.062	0.031	0.016
165	0.078	0.031	0.047	0.016
166	0.047	0.031	0.016	0.031
167	0.062	0.031	0.016	0.016
168	0.047	0.031	0.016	0.031
169	0.047	0.031	0.016	0.031
170	0.047	0.031	0.016	0.047
171	0.047	0.031	0.016	0.016
172	0.062	0.031	0.000	0.031
173	0.078	0.031	0.016	0.016
174	0.031	0.031	0.000	0.047
175	0.062	0.031	0.047	0.016

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
176	0.031	0.031	0.016	0.016
177	0.062	0.031	0.016	0.031
178	0.031	0.031	0.016	0.047
179	0.062	0.031	0.016	0.016
180	0.031	0.031	0.016	0.016
181	0.031	0.031	0.016	0.031
182	0.062	0.031	0.016	0.031
183	0.078	0.031	0.016	0.031
184	0.047	0.031	0.016	0.031
185	0.031	0.031	0.016	0.031
186	0.062	0.031	0.016	0.000
187	0.062	0.016	0.016	0.000
188	0.031	0.016	0.016	0.031
189	0.062	0.016	0.016	0.000
190	0.047	0.016	0.016	0.031
191	0.047	0.016	0.000	0.031
192	0.031	0.016	0.031	0.016
193	0.047	0.000	0.016	0.000
194	0.047	0.031	0.016	0.031
195	0.031	0.016	0.000	0.031
196	0.047	0.016	0.016	0.016
197	0.047	0.000	0.016	0.000
198	0.031	0.031	0.000	0.031
199	0.062	0.031	0.016	0.031
200	0.031	0.016	0.016	0.016

A.5 P=[10,0;20,0;30,10;30,20;20,30;10,30;0,20;0,10;10,0]

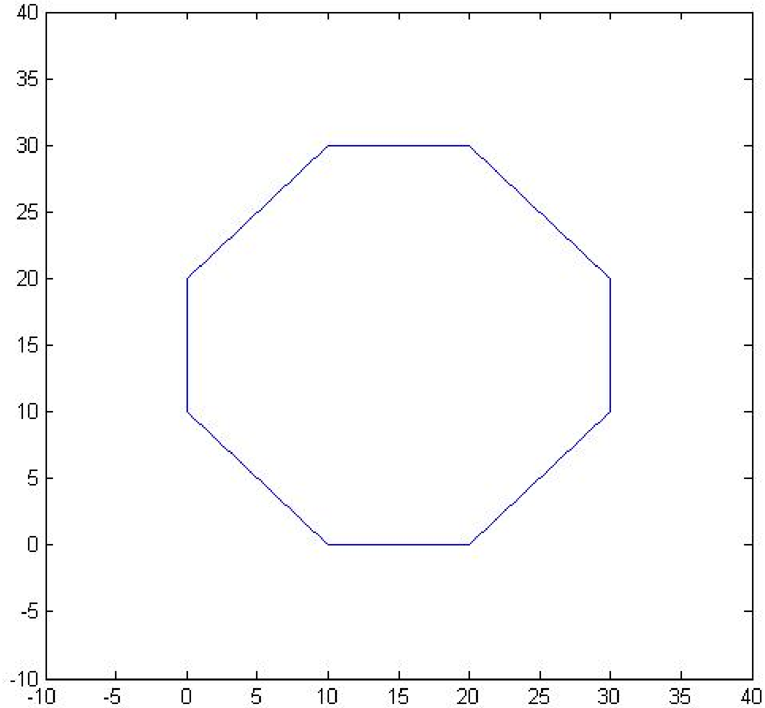


Figure 35 P=[10,0;20,0;30,10;30,20;20,30;10,30;0,20;0,10;10,0]

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
1	894	516	728	707	516	223	2.31
2	220	132	148	166	132	56	2.36
3	107	68	72	80	68	25	2.72
4	51	36	45	46	36	14	2.57
5	35	24	30	33	24	9	2.67
6	26	16	22	22	16	7	2.29
7	19	16	16	16	16	5	3.20
8	14	16	16	12	12	4	3.00
9	11	8	8	10	8	3	2.67
10	8	8	8	8	8	3	2.67
11	7	4	4	4	4	2	2.00
12	6	4	4	4	4	2	2.00
13	4	4	4	4	4	2	2.00
14	4	4	4	4	4	2	2.00
15	4	4	4	4	4	1	4.00
16	1	1	1	1	1	1	1.00
17	1	1	1	1	1	1	1.00
18	1	1	1	1	1	1	1.00
19	1	1	1	1	1	1	1.00
20	1	1	1	1	1	1	1.00

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
7.76885	1.887612	2.761218	2.667617	
1.872012	0.452403	0.514803	0.624004	
0.920406	0.249602	0.327602	0.296402	
0.452403	0.140401	0.187201	0.187201	
0.312002	0.093601	0.109201	0.124801	
0.234002	0.0468	0.078	0.078	
0.171601	0.078001	0.0468	0.0624	
0.124801	0.078001	0.093601	0.0312	
0.124801	0.0312	0.0156	0.0624	
0.093601	0.0312	0.0468	0.0312	
0.0624	0	0.0312	0.0156	
0.078001	0.0312	0.0468	0.0156	
0.0468	0.0312	0.0156	0.0156	
0.0624	0.0312	0.0156	0	
0.0312	0.0312	0.0156	0.0312	
0.0312	0	0	0	
0	0	0	0	
0	0	0	0	
0	0	0.0312	0.0156	
0	0	0	0	

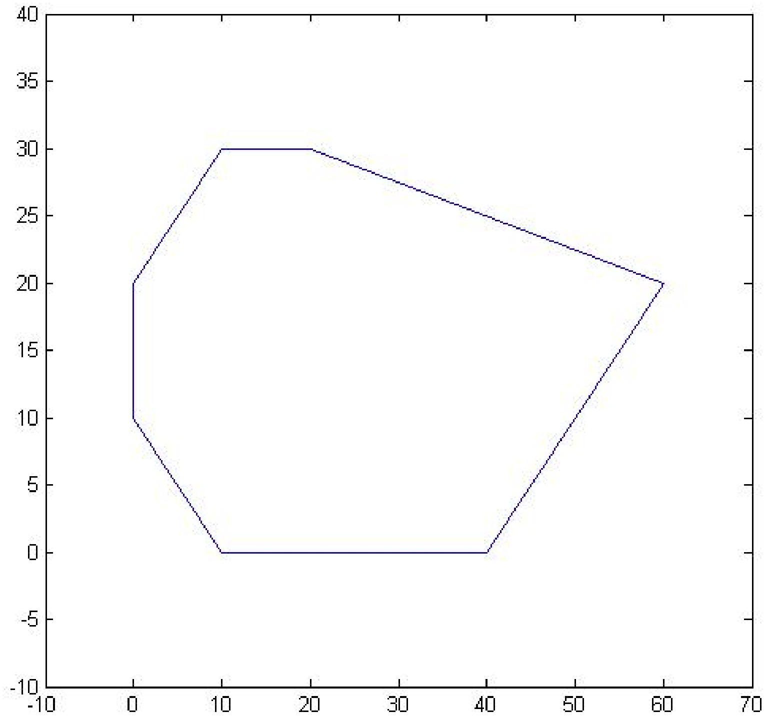


Figure 36 P=[10,0;40,0;50,10;60,20;20,30;10,30;0,20;0,10;10,0]

A.6 P=[10,0;40,0;50,10;60,20;20,30;10,30;0,20;0,10;10,0]

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
1	1653	995	1304	1327	995	414	2.40
2	412	254	299	321	254	104	2.44
3	187	127	129	147	127	46	2.76
4	97	67	78	87	67	26	2.58
5	64	43	51	57	43	17	2.53
6	41	33	35	42	33	12	2.75
7	33	28	27	30	27	9	3.00
8	23	21	21	22	21	7	3.00
9	19	16	16	17	16	6	2.67
10	14	13	15	13	13	5	2.60

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
11	11	9	12	12	9	4	2.25
12	11	8	10	11	8	3	2.67
13	11	8	9	9	8	3	2.67
14	8	8	9	6	6	3	2.00
15	8	8	6	6	6	2	3.00
16	6	6	5	6	5	2	2.50
17	6	4	4	5	4	2	2.00
18	4	4	4	5	4	2	2.00
19	4	4	4	3	3	2	1.50
20	4	3	3	3	3	2	1.50

Computation Time (Sec)

β	**K and A**	**MOPAD1**	**MOPAD2**	**MOPAD3**
14.758	3.728	5.117	5.273	
3.526	0.874	1.092	1.170	
1.638	0.437	0.452	0.530	
0.811	0.265	0.281	0.312	
0.530	0.140	0.203	0.234	
0.390	0.140	0.125	0.140	
0.296	0.094	0.125	0.125	
0.203	0.094	0.078	0.125	
0.203	0.047	0.047	0.078	
0.125	0.062	0.078	0.062	
0.140	0.062	0.047	0.062	
0.094	0.016	0.031	0.062	
0.125	0.047	0.047	0.031	
0.078	0.047	0.062	0.031	
0.094	0.047	0.016	0.016	
0.078	0.047	0.031	0.031	
0.062	0.016	0.031	0.047	
0.031	0.016	0.031	0.016	
0.062	0.031	0.031	0.031	
0.031	0.016	0.031	0.000	

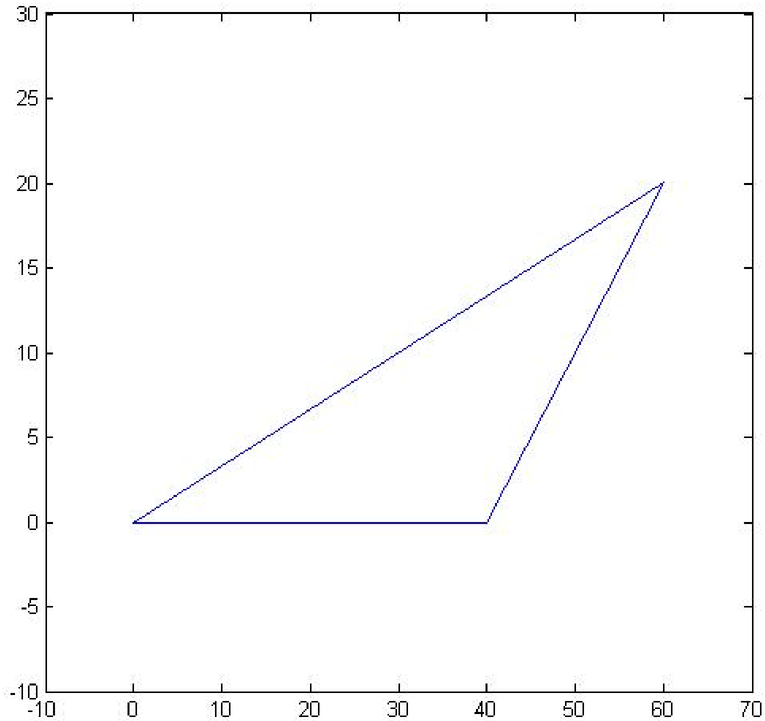


Figure 37   P=[0,0;60,0;40,0;0,0]

A.7   P=[0,0;60,0;40,0;0,0]

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
1	501	327	417	431	327	128	2.55
2	132	91	101	116	91	32	2.84
3	56	45	48	56	45	15	3.00
4	32	30	30	30	30	8	3.75
5	21	16	20	21	16	6	2.67
6	14	14	15	16	14	4	3.50
7	11	14	12	10	10	3	3.33
8	11	8	10	9	8	2	4.00
9	7	7	7	8	7	2	3.50
10	6	7	7	7	7	2	3.50

Required Guards

β	K and A	MOPAD1	MOPAD2	MOPAD3	MOPAD	ω	MOPADω
11	7	5	6	6	5	2	2.50
12	4	4	6	6	4	1	4.00
13	4	4	5	5	4	1	4.00
14	4	4	5	5	4	1	4.00
15	4	4	4	4	4	1	4.00
16	4	4	4	4	4	1	4.00
17	4	4	4	3	3	1	3.00
18	4	4	4	3	3	1	3.00
19	2	2	2	3	2	1	2.00
20	2	2	2	3	2	1	2.00

Computation Time (Sec)

β	K and A	MOPAD1	MOPAD2	MOPAD3
4.415	1.170	1.747	1.622	
1.123	0.328	0.343	0.437	
0.484	0.156	0.172	0.203	
0.265	0.094	0.125	0.109	
0.172	0.078	0.094	0.078	
0.094	0.047	0.047	0.047	
0.094	0.062	0.047	0.062	
0.109	0.047	0.047	0.031	
0.047	0.047	0.047	0.031	
0.031	0.016	0.031	0.016	
0.078	0.031	0.016	0.047	
0.016	0.031	0.047	0.016	
0.047	0.031	0.016	0.047	
0.047	0.031	0.016	0.016	
0.031	0.031	0.016	0.031	
0.031	0.031	0.031	0.016	
0.016	0.016	0.016	0.031	
0.047	0.016	0.016	0.000	
0.016	0.000	0.000	0.016	
0.000	0.031	0.031	0.031	

A.8 900 Independent Randomly Generated P and β

The data in this table was generated by implementing the MOPAD method on 900 independent and randomly generated polygons with independent randomly generated visibility ranges. The column " Vertices" is the number of vertices in P.

vertices	ω	**MOPAD**	**MOPAD** ω
3	1	4	4.00
3	1	4	4.00
3	1	8	8.00
3	1	2	2.00
3	1	2	2.00
3	1	4	4.00
3	1	1	1.00
3	1	4	4.00
3	1	5	5.00
3	1	4	4.00
3	1	4	4.00
3	1	8	8.00
3	1	1	1.00
3	1	8	8.00
3	1	2	2.00
3	1	4	4.00
3	1	4	4.00
3	1	1	1.00
4	1	1	1.00
4	1	2	2.00
4	1	2	2.00
4	1	4	4.00
4	1	3	3.00
4	1	4	4.00
4	1	4	4.00
4	1	2	2.00
5	1	1	1.00
5	1	2	2.00

vertices	ω	**MOPAD**	**MOPAD** ω
3	2	7	3.50
3	2	8	4.00
3	2	6	3.00
3	2	7	3.50
3	2	6	3.00
3	2	7	3.50
3	2	9	4.50
3	2	8	4.00
3	2	16	8.00
3	2	7	3.50
3	2	7	3.50
3	2	6	3.00
4	2	7	3.50
4	2	6	3.00
4	2	4	2.00
4	2	5	2.50
4	2	8	4.00
4	2	8	4.00
4	2	9	4.50
4	2	8	4.00
4	2	8	4.00
5	2	8	4.00
5	2	4	2.00
5	2	5	2.50
5	2	6	3.00
5	2	7	3.50
5	2	8	4.00
5	2	7	3.50

vertices	ω	**MOPAD**	**MOPAD** ω
5	2	4	2.00
5	2	6	3.00
6	2	8	4.00
6	2	8	4.00
6	2	6	3.00
6	2	8	4.00
7	2	7	3.50
7	2	8	4.00
7	2	8	4.00
7	2	4	2.00
8	2	7	3.50
8	2	8	4.00
8	2	8	4.00
8	2	8	4.00
8	2	7	3.50
9	2	7	3.50
9	2	6	3.00
9	2	7	3.50
9	2	5	2.50
11	2	8	4.00
11	2	7	3.50
13	2	6	3.00
13	2	7	3.50
17	2	7	3.50
3	3	12	4.00
3	3	8	2.67
3	3	9	3.00
3	3	10	3.33

vertices	ω	**MOPAD**	**MOPAD** ω
3	3	10	3.33
4	3	12	4.00
4	3	8	2.67
4	3	20	6.67
4	3	8	2.67
4	3	10	3.33
4	3	9	3.00
4	3	9	3.00
4	3	8	2.67
5	3	8	2.67
5	3	12	4.00
5	3	8	2.67
5	3	8	2.67
5	3	9	3.00
5	3	8	2.67
6	3	5	1.67
6	3	8	2.67
6	3	8	2.67
6	3	8	2.67
6	3	8	2.67
6	3	7	2.33
6	3	9	3.00
6	3	8	2.67
7	3	8	2.67
7	3	8	2.67
7	3	8	2.67
7	3	8	2.67
7	3	8	2.67

vertices	ω	**MOPAD**	**MOPAD** ω
7	3	10	3.33
7	3	9	3.00
8	3	8	2.67
8	3	8	2.67
8	3	8	2.67
8	3	8	2.67
8	3	8	2.67
9	3	8	2.67
9	3	8	2.67
9	3	10	3.33
9	3	8	2.67
9	3	7	2.33
9	3	8	2.67
9	3	8	2.67
9	3	9	3.00
9	3	8	2.67
10	3	8	2.67
10	3	8	2.67
10	3	8	2.67
10	3	9	3.00
10	3	8	2.67
10	3	8	2.67
10	3	10	3.33
10	3	10	3.33
10	3	11	3.67
10	3	9	3.00
10	3	8	2.67
11	3	8	2.67

vertices	ω	**MOPAD**	**MOPAD** ω
11	3	8	2.67
11	3	8	2.67
11	3	8	2.67
11	3	8	2.67
12	3	8	2.67
12	3	9	3.00
12	3	8	2.67
12	3	8	2.67
13	3	8	2.67
13	3	8	2.67
13	3	8	2.67
13	3	8	2.67
13	3	9	3.00
13	3	10	3.33
13	3	9	3.00
13	3	8	2.67
13	3	8	2.67
13	3	8	2.67
13	3	8	2.67
14	3	8	2.67
14	3	8	2.67
14	3	7	2.33
14	3	8	2.67
14	3	8	2.67
14	3	9	3.00
14	3	9	3.00
14	3	8	2.67
14	3	8	2.67

vertices	ω	**MOPAD**	**MOPAD** ω
14	3	8	2.67
15	3	8	2.67
15	3	9	3.00
15	3	10	3.33
15	3	10	3.33
15	3	10	3.33
15	3	8	2.67
15	3	8	2.67
16	3	8	2.67
16	3	8	2.67
16	3	8	2.67
16	3	8	2.67
16	3	8	2.67
16	3	8	2.67
17	3	8	2.67
17	3	8	2.67
17	3	8	2.67
17	3	8	2.67
17	3	8	2.67
18	3	8	2.67
18	3	8	2.67
18	3	8	2.67
18	3	8	2.67
18	3	8	2.67
19	3	8	2.67
19	3	9	3.00
19	3	9	3.00
19	3	8	2.67

vertices	ω	**MOPAD**	**MOPAD** ω
19	3	8	2.67
19	3	8	2.67
19	3	8	2.67
19	3	8	2.67
19	3	8	2.67
19	3	8	2.67
20	3	8	2.67
20	3	8	2.67
20	3	8	2.67
20	3	8	2.67
20	3	9	3.00
20	3	10	3.33
20	3	8	2.67
20	3	8	2.67
3	4	14	3.50
3	4	13	3.25
4	4	16	4.00
4	4	16	4.00
4	4	13	3.25
4	4	13	3.25
5	4	8	2.00
5	4	11	2.75
5	4	11	2.75
5	4	13	3.25
6	4	11	2.75
6	4	11	2.75
6	4	14	3.50
6	4	12	3.00

vertices	ω	**MOPAD**	**MOPAD** ω
6	4	13	3.25
7	4	13	3.25
7	4	16	4.00
7	4	12	3.00
7	4	12	3.00
7	4	14	3.50
8	4	14	3.50
8	4	12	3.00
8	4	15	3.75
8	4	12	3.00
9	4	12	3.00
9	4	9	2.25
9	4	11	2.75
9	4	12	3.00
9	4	16	4.00
10	4	12	3.00
10	4	13	3.25
10	4	9	2.25
10	4	8	2.00
10	4	15	3.75
10	4	12	3.00
10	4	14	3.50
10	4	16	4.00
10	4	12	3.00
11	4	14	3.50
12	4	16	4.00
12	4	12	3.00
12	4	11	2.75

vertices	ω	MOPAD	MOPAD ω
12	4	8	2.00
12	4	8	2.00
12	4	13	3.25
12	4	11	2.75
13	4	12	3.00
13	4	12	3.00
13	4	13	3.25
13	4	13	3.25
13	4	10	2.50
13	4	12	3.00
13	4	11	2.75
13	4	12	3.00
13	4	10	2.50
13	4	14	3.50
14	4	10	2.50
14	4	8	2.00
14	4	11	2.75
14	4	8	2.00
14	4	10	2.50
14	4	9	2.25
14	4	14	3.50
15	4	8	2.00
15	4	15	3.75
15	4	11	2.75
15	4	9	2.25
15	4	13	3.25
15	4	16	4.00
15	4	11	2.75

vertices	ω	**MOPAD**	**MOPAD** ω
15	4	10	2.50
16	4	11	2.75
16	4	12	3.00
16	4	12	3.00
16	4	13	3.25
17	4	12	3.00
17	4	11	2.75
17	4	12	3.00
17	4	10	2.50
17	4	9	2.25
18	4	15	3.75
18	4	11	2.75
18	4	10	2.50
18	4	14	3.50
18	4	14	3.50
18	4	12	3.00
18	4	13	3.25
19	4	14	3.50
19	4	12	3.00
20	4	10	2.50
20	4	13	3.25
20	4	8	2.00
20	4	8	2.00
20	4	9	2.25
5	5	12	2.40
5	5	16	3.20
5	5	16	3.20
5	5	14	2.80

vertices	ω	**MOPAD**	**MOPAD** ω
5	5	16	3.20
7	5	16	3.20
7	5	16	3.20
7	5	13	2.60
7	5	16	3.20
8	5	16	3.20
8	5	16	3.20
8	5	16	3.20
8	5	16	3.20
9	5	16	3.20
9	5	13	2.60
9	5	16	3.20
9	5	16	3.20
10	5	13	2.60
10	5	16	3.20
10	5	12	2.40
10	5	16	3.20
10	5	16	3.20
11	5	16	3.20
11	5	16	3.20
11	5	16	3.20
11	5	12	2.40
11	5	12	2.40
11	5	12	2.40
12	5	15	3.00
12	5	14	2.80
12	5	16	3.20
13	5	16	3.20

vertices	ω	**MOPAD**	**MOPAD** ω
13	5	16	3.20
13	5	12	2.40
13	5	16	3.20
13	5	16	3.20
14	5	13	2.60
14	5	16	3.20
15	5	16	3.20
15	5	12	2.40
16	5	16	3.20
16	5	12	2.40
17	5	16	3.20
17	5	16	3.20
17	5	12	2.40
17	5	16	3.20
17	5	16	3.20
18	5	16	3.20
18	5	16	3.20
19	5	15	3.00
19	5	16	3.20
19	5	12	2.40
19	5	16	3.20
20	5	16	3.20
20	5	12	2.40
3	6	20	3.33
6	6	16	2.67
7	6	16	2.67
7	6	17	2.83
7	6	17	2.83

vertices	ω	**MOPAD**	**MOPAD** ω
8	6	18	3.00
9	6	16	2.67
9	6	16	2.67
9	6	17	2.83
10	6	17	2.83
10	6	16	2.67
10	6	16	2.67
11	6	16	2.67
11	6	16	2.67
11	6	16	2.67
12	6	15	2.50
12	6	16	2.67
12	6	16	2.67
13	6	16	2.67
13	6	16	2.67
14	6	16	2.67
15	6	13	2.17
15	6	16	2.67
16	6	17	2.83
16	6	16	2.67
16	6	13	2.17
16	6	14	2.33
17	6	18	3.00
17	6	16	2.67
17	6	16	2.67
18	6	16	2.67
18	6	16	2.67
19	6	12	2.00

vertices	ω	**MOPAD**	**MOPAD** ω
19	6	16	2.67
19	6	13	2.17
20	6	16	2.67
20	6	16	2.67
20	6	12	2.00
20	6	16	2.67
4	7	20	2.86
4	7	15	2.14
5	7	26	3.71
5	7	22	3.14
6	7	19	2.71
6	7	19	2.71
8	7	18	2.57
8	7	18	2.57
8	7	16	2.29
8	7	21	3.00
9	7	17	2.43
9	7	18	2.57
9	7	16	2.29
10	7	20	2.86
10	7	17	2.43
11	7	21	3.00
11	7	19	2.71
11	7	18	2.57
12	7	16	2.29
12	7	16	2.29
12	7	19	2.71
12	7	17	2.43

vertices	ω	**MOPAD**	**MOPAD** ω
13	7	21	3.00
14	7	17	2.43
14	7	21	3.00
14	7	17	2.43
15	7	17	2.43
15	7	19	2.71
16	7	17	2.43
16	7	15	2.14
17	7	18	2.57
17	7	17	2.43
18	7	17	2.43
19	7	17	2.43
20	7	15	2.14
4	8	23	2.88
4	8	23	2.88
5	8	28	3.50
6	8	27	3.38
6	8	23	2.88
6	8	21	2.63
7	8	25	3.13
7	8	19	2.38
7	8	24	3.00
7	8	23	2.88
8	8	18	2.25
8	8	22	2.75
8	8	19	2.38
9	8	21	2.63
9	8	23	2.88

vertices	ω	**MOPAD**	**MOPAD** ω
9	8	26	3.25
9	8	21	2.63
10	8	19	2.38
11	8	17	2.13
11	8	20	2.50
12	8	21	2.63
12	8	19	2.38
12	8	23	2.88
13	8	19	2.38
14	8	18	2.25
15	8	21	2.63
15	8	21	2.63
16	8	17	2.13
16	8	21	2.63
16	8	21	2.63
17	8	22	2.75
18	8	17	2.13
19	8	19	2.38
20	8	19	2.38
3	9	33	3.67
5	9	29	3.22
5	9	26	2.89
5	9	28	3.11
5	9	33	3.67
6	9	30	3.33
7	9	25	2.78
7	9	24	2.67
8	9	19	2.11

vertices	ω	**MOPAD**	**MOPAD** ω
8	9	28	3.11
9	9	26	2.89
9	9	29	3.22
10	9	24	2.67
10	9	22	2.44
10	9	27	3.00
11	9	24	2.67
11	9	20	2.22
11	9	28	3.11
12	9	25	2.78
12	9	28	3.11
12	9	26	2.89
14	9	23	2.56
14	9	27	3.00
16	9	25	2.78
16	9	22	2.44
17	9	22	2.44
17	9	26	2.89
17	9	28	3.11
17	9	18	2.00
17	9	29	3.22
18	9	25	2.78
18	9	26	2.89
19	9	23	2.56
20	9	25	2.78
20	9	25	2.78
5	10	32	3.20
6	10	29	2.90

vertices	ω	**MOPAD**	**MOPAD** ω
6	10	33	3.30
9	10	30	3.00
11	10	32	3.20
11	10	32	3.20
12	10	23	2.30
13	10	30	3.00
14	10	22	2.20
15	10	21	2.10
15	10	29	2.90
15	10	30	3.00
15	10	21	2.10
20	10	21	2.10
20	10	30	3.00
4	11	25	2.27
7	11	25	2.27
9	11	31	2.82
9	11	32	2.91
12	11	34	3.09
12	11	32	2.91
13	11	33	3.00
15	11	32	2.91
16	11	32	2.91
16	11	32	2.91
16	11	31	2.82
18	11	32	2.91
19	11	32	2.91
20	11	32	2.91
3	12	30	2.50

vertices	ω	**MOPAD**	**MOPAD** ω
4	12	43	3.58
6	12	32	2.67
6	12	31	2.58
6	12	26	2.17
6	12	34	2.83
6	12	24	2.00
7	12	37	3.08
11	12	36	3.00
11	12	33	2.75
11	12	32	2.67
15	12	35	2.92
17	12	34	2.83
18	12	27	2.25
19	12	33	2.75
20	12	32	2.67
5	13	37	2.85
5	13	42	3.23
6	13	33	2.54
7	13	32	2.46
8	13	36	2.77
8	13	33	2.54
13	13	33	2.54
15	13	32	2.46
16	13	36	2.77
17	13	34	2.62
17	13	23	1.77
3	14	41	2.93
6	14	42	3.00

vertices	ω	**MOPAD**	**MOPAD** ω
7	14	34	2.43
9	14	38	2.71
11	14	37	2.64
11	14	35	2.50
11	14	35	2.50
13	14	39	2.79
17	14	37	2.64
17	14	39	2.79
19	14	34	2.43
19	14	26	1.86
19	14	37	2.64
20	14	37	2.64
20	14	36	2.57
6	15	27	1.80
6	15	36	2.40
7	15	37	2.47
10	15	45	3.00
11	15	36	2.40
13	15	41	2.73
17	15	45	3.00
17	15	39	2.60
17	15	37	2.47
18	15	39	2.60
18	15	44	2.93
20	15	37	2.47
8	16	48	3.00
9	16	32	2.00
11	16	27	1.69

vertices	ω	**MOPAD**	**MOPAD** ω
11	16	44	2.75
14	16	47	2.94
15	16	29	1.81
17	16	40	2.50
18	16	28	1.75
18	16	44	2.75
20	16	27	1.69
3	17	39	2.29
4	17	34	2.00
6	17	30	1.76
6	17	32	1.88
12	17	54	3.18
13	17	51	3.00
14	17	52	3.06
19	17	30	1.76
3	18	52	2.89
4	18	55	3.06
5	18	56	3.11
6	18	53	2.94
8	18	52	2.89
11	18	52	2.89
12	18	59	3.28
14	18	28	1.56
14	18	31	1.72
15	18	57	3.17
18	18	31	1.72
18	18	56	3.11
20	18	56	3.11

vertices	ω	**MOPAD**	**MOPAD** ω
5	19	57	3.00
7	19	62	3.26
11	19	34	1.79
11	19	37	1.95
13	19	30	1.58
14	19	62	3.26
19	19	61	3.21
19	19	58	3.05
20	19	61	3.21
8	20	64	3.20
8	20	62	3.10
8	20	61	3.05
12	20	63	3.15
12	20	33	1.65
12	20	62	3.10
13	20	61	3.05
18	20	31	1.55
18	20	39	1.95
20	20	63	3.15
8	21	39	1.86
13	21	63	3.00
17	21	63	3.00
17	21	63	3.00
18	21	38	1.81
4	22	41	1.86
6	22	63	2.86
9	22	66	3.00
9	22	63	2.86

vertices	ω	**MOPAD**	**MOPAD** ω
15	22	64	2.91
5	23	67	2.91
7	23	66	2.87
12	23	66	2.87
17	23	64	2.78
18	23	64	2.78
19	23	64	2.78
19	23	64	2.78
4	24	65	2.71
7	24	66	2.75
10	24	66	2.75
14	24	64	2.67
14	24	65	2.71
15	24	64	2.67
18	24	65	2.71
7	25	64	2.56
9	25	65	2.60
18	25	34	1.36
20	25	64	2.56
4	26	68	2.62
8	26	38	1.46
10	26	67	2.58
14	26	64	2.46
14	26	64	2.46
15	26	38	1.46
16	26	64	2.46
9	27	67	2.48
12	27	44	1.63

vertices	ω	**MOPAD**	**MOPAD** ω
16	27	38	1.41
18	27	36	1.33
6	28	82	2.93
6	28	77	2.75
8	28	69	2.46
11	28	69	2.46
16	28	65	2.32
16	28	66	2.36
19	28	69	2.46
8	29	71	2.45
14	29	71	2.45
20	29	75	2.59
6	30	85	2.83
7	30	85	2.83
8	30	44	1.47
5	31	79	2.55
14	31	45	1.45
15	31	46	1.48
16	31	77	2.48
18	31	77	2.48
19	31	80	2.58
18	33	81	2.45
17	34	46	1.35
18	34	84	2.47
15	35	46	1.31
6	36	98	2.72
10	36	50	1.39
10	36	91	2.53

vertices	ω	**MOPAD**	**MOPAD** ω
13	36	56	1.56
14	36	95	2.64
16	36	90	2.50
18	36	53	1.47
17	37	94	2.54
20	37	51	1.38
15	38	49	1.29
12	39	117	3.00
16	40	124	3.10
16	41	120	2.93
18	42	127	3.02
12	43	126	2.93
19	43	127	2.95
20	43	127	2.95
16	44	127	2.89
5	45	125	2.78
7	45	125	2.78
7	45	54	1.20
8	45	129	2.87
17	45	130	2.89
18	45	50	1.11
19	45	127	2.82
6	46	114	2.48
15	46	128	2.78
15	46	128	2.78
15	46	131	2.85
8	47	129	2.74
11	47	132	2.81

vertices	ω	**MOPAD**	**MOPAD** ω
19	47	127	2.70
8	48	129	2.69
6	49	135	2.76
11	49	128	2.61
14	50	131	2.62
16	50	129	2.58
18	50	129	2.58
16	52	131	2.52
20	52	131	2.52
6	53	136	2.57
6	53	147	2.77
13	53	136	2.57
18	53	55	1.04
5	54	138	2.56
18	55	56	1.02
5	57	148	2.60
7	58	143	2.47
15	58	137	2.36
8	60	149	2.48
12	60	144	2.40
19	60	147	2.45
4	61	143	2.34
17	61	147	2.41
6	63	142	2.25
19	63	146	2.32
6	64	178	2.78
7	64	163	2.55
10	64	159	2.48

vertices	ω	**MOPAD**	**MOPAD** ω
11	64	160	2.50
5	65	174	2.68
8	65	150	2.31
8	67	190	2.84
8	67	68	1.01
10	67	77	1.15
14	67	192	2.87
16	67	175	2.61
14	68	198	2.91
11	69	209	3.03
13	69	74	1.07
17	69	198	2.87
20	69	204	2.96
6	71	217	3.06
20	71	215	3.03
4	72	197	2.74
16	72	226	3.14
5	75	93	1.24
9	76	217	2.86
16	76	83	1.09
14	78	227	2.91
15	80	246	3.08
10	85	252	2.96
16	85	253	2.98
18	86	123	1.43
9	87	96	1.10
12	87	253	2.91
14	88	250	2.84

vertices	ω	MOPAD	MOPAD ω
12	89	103	1.16
16	96	115	1.20
14	97	257	2.65
18	97	257	2.65
20	98	257	2.62
7	99	258	2.61
19	99	154	1.56
15	100	163	1.63
14	101	257	2.54
19	101	258	2.55
16	102	258	2.53
4	104	296	2.85
7	104	260	2.50
15	105	265	2.52
3	106	308	2.91
11	108	266	2.46
19	109	264	2.42
7	110	261	2.37
20	114	266	2.33
16	117	274	2.34
3	120	289	2.41
18	120	279	2.33
18	124	175	1.41
20	124	165	1.33
3	129	205	1.59
12	130	298	2.29
11	133	302	2.27
19	133	321	2.41

vertices	ω	**MOPAD**	**MOPAD** ω
20	138	300	2.17
17	141	323	2.29
19	142	303	2.13
17	145	357	2.46
4	150	404	2.69
10	152	406	2.67
17	153	417	2.73
15	154	472	3.06
20	154	369	2.40
10	156	315	2.02
9	158	446	2.82
13	162	484	2.99
13	166	496	2.99
19	183	500	2.73
5	185	362	1.96
12	188	506	2.69
16	192	412	2.15
10	205	508	2.48
5	206	508	2.47
12	206	505	2.45
4	211	221	1.05
14	216	367	1.70
8	220	528	2.40
16	221	403	1.82
13	222	518	2.33
11	230	543	2.36
11	236	502	2.13
13	248	531	2.14

vertices	ω	**MOPAD**	**MOPAD** ω
7	259	652	2.52
20	265	761	2.87
9	268	700	2.61
19	271	819	3.02
16	274	774	2.82
4	279	714	2.56
11	330	945	2.86
3	331	904	2.73
3	333	897	2.69
13	357	1000	2.80
19	357	1019	2.85
19	362	467	1.29
7	390	1002	2.57
10	420	976	2.32
3	424	617	1.46
14	482	1040	2.16
9	483	1067	2.21
10	484	1068	2.21
20	494	1065	2.16
15	512	1080	2.11
10	520	1179	2.27
7	592	1633	2.76
16	596	781	1.31
17	615	1579	2.57
16	632	956	1.51
5	649	1728	2.66
9	665	1880	2.83
12	717	1056	1.47

vertices	ω	**MOPAD**	**MOPAD** ω
15	872	2004	2.30
5	954	2313	2.42
20	1062	2479	2.33
12	1200	3541	2.95
13	1428	1567	1.10
15	1461	3957	2.71
4	1546	3645	2.36
14	1683	1985	1.18
9	1696	2036	1.20
17	1717	4086	2.38
4	1726	3916	2.27
19	1754	3154	1.80
14	2033	4972	2.45
12	2065	4466	2.16
12	2091	5182	2.48
4	2845	4568	1.61
17	3121	7900	2.53
3	3124	7216	2.31
18	3763	8443	2.24
5	3944	10520	2.67
19	4031	8461	2.10
7	4416	10639	2.41
11	5054	14103	2.79
8	5111	15103	2.95
8	5274	14248	2.70
15	5697	6001	1.05
4	8286	19655	2.37
18	9297	21312	2.29

vertices	ω	**MOPAD**	**MOPAD** ω
10	13098	32046	2.45
10	13929	32299	2.32
20	17020	44032	2.59
12	19535	52123	2.67

Appendix B. 25 Key Cities

	Metro Area	Latitude	Longitude
1	New York, NY	40.38.22.400 N	073.45.59.200 W
2	Los Angeles, CA	33.55.56.600 N	118.24.25.000 W
3	Chicago, IL	41.58.49.100 N	087.55.42.000 W
4	Houston, TX	29.59.53.800 N	095.21.12.800 W
5	Philadelphia, PA	39.51.32.700 N	075.16.00.200 W
6	Phoenix, AZ	33.25.37.200 N	112.00.23.400 W
7	San Antonio, TX	29.33.32.200 N	098.28.09.400 W
8	San Diego, CA	32.52.59.800 N	117.08.37.800 W
9	Dallas, TX	32.55.20.500 N	097.02.37.800 W
10	San Jose, CA	37.25.28.300 N	122.00.54.400 W
11	Detroit, MI	42.12.47.200 N	083.22.44.800 W
12	Indianapolis, IN	39.42.08.800 N	086.17.19.800 W
13	Jacksonville, FL	30.29.36.000 N	081.41.33.600 W
14	San Francisco, CA	37.42.22.100 N	122.13.31.200 W
15	Columbus, OH	40.00.28.200 N	082.53.39.600 W
16	Orlando, FL	28.23.41.400 N	081.18.16.700 W
17	Seattle, WA	47.27.09.400 N	122.19.01.700 W
18	Boston, MA	42.20.54.800 N	071.00.21.900 W
19	Denver, CO	39.45.38.200 N	104.52.26.900 W
20	Washington, DC	38.50.42.700 N	077.02.00.400 W
21	Las Vegas, NV	36.05.00.400 N	115.09.36.100 W
22	Atlanta, GA	33.53.39.200 N	084.29.54.900 W
23	Virginia Beach, VA	36.49.38.700 N	076.00.49.600 W
24	Miami, FL	25.47.52.000 N	080.17.36.300 W
25	Minneapolis, MN	44.53.25.400 N	093.13.50.300 W

Determining Locations of Multi-Function Phased Array Radar

— BLUE DART —

Kassandra M. Merritt, Capt, USAF*

13 June 2011

Air surveillance of United States territory is an essential Department of Defense (DoD) function. In the event of an attack on North America by an incoming aerial threat such as a hijacked or enemy airplane, missile, or any other threat to national security, the surveillance capabilities of the DoD, Department of Homeland Security (DHS), and Federal Aviation Administration (FAA) are critical to discovering and tracking the threat so that the DoD can eliminate it. The current National Airspace System (NAS) provides coverage from the surface to 60,000 feet mean sea level (MSL) using primary and secondary FAA long and short range radars, defense radars, and additional surveillance systems along the borders and other areas of interest. The current radar system consists of weather and aerial surveillance radars that operate by using a rotating antenna to sweep a large area. Many of these radars are reaching the end of their design life within the next ten to twenty years. Additionally, the current surveillance system has significant surveillance gaps at low altitude and is limited in its ability to detect objects with small radar cross sections such as small missiles. The Multi-Function

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Phased Array Radar (MPAR) is capable of performing several missions including weather and aerial surveillance with one unit. Additionally, MPAR units provide increased detection capabilities for small radar cross section objects and increased reliability. By replacing the current radar network with a single integrated network of MPAR units, surveillance capabilities can be greatly enhanced and life cycle cost can be reduced.

The question of how to optimally use a limited number of resources is not new. Researchers have tried to answer this question in several different forms. Many of these traditional models of the problem are limited in their scope and or scale and are therefore not able to handle the MPAR location problem. Additionally, because of the size of the area of desired coverage and budget restrictions, full coverage of the territory at low altitude may not be possible. There are currently no methods available for determining where to locate the MPAR units in order to provide some level of coverage less than 100 percent.

The developed methodology can be used to determine a small number and location of MPAR units required to cover any given region. Specifically, the methodology treats the region of surveillance as a polygon and the MPAR units as guards with a circular area of visibility with a constant range. The method can determine this small set of MPAR units given a requirement for full coverage or for any percentage between 0 and 100. This methodology can also be used to provide recommendations for surveillance over key areas or events, placement of communications resources or other limited range resources with unconstrained available locations at a reduced cost.

The current age of the NAS and availability of new technology requires that the system be updated in the near future. Because of the size of this system and the cost associated with each unit, it is important to investigate how to place these units and how many will be required to attain acceptable coverage. Current methodologies are limited in their ability to handle problems of this magnitude and are unable to consider less than 100 percent coverage. The methodology developed as a result of this research can be used to provide insight for the acquisitions process and options for placing units under budgetary or operational constraints that result in a need for less than 100 percent coverage.

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14. ABSTRACT
Air surveillance of United States territory is an essential Department of Defense function. In the event of an incoming aerial attack on North America, the DoD, Department of Homeland Security, and Federal Aviation Administration surveillance capabilities are critical to discovering and tracking the threat so that it can be eliminated. Many of the currently used surveillance radar will reach the end of their design life within ten to twenty years. By replacing the current radar network with a single integrated network of Multifunction Phased Array Radar (MPAR) units, surveillance capabilities can be enhanced and life cycle cost can be reduced. The problem of determining the location of required MPAR units to provide sufficient air surveillance of a given area is a large problem that could require a prohibitively long time to solve. By representing the area of surveillance as a polygon and the MPAR units as guards with a defined circle of detection, this problem as well as similar surveillance or coverage problems can be expressed with easily adjustable parameters. The problem of covering the interior and exterior of a polygon region with a minimal number of guards with homogeneous capabilities is not well researched. There are no methods for determining the minimal number of guards required to cover the interior and exterior of a polygon at a desired coverage level less than 100 percent. This paper describes an iterative method for determining a small number and location of guards required to cover a convex polygon both fully and at a specified percentage coverage less than 100 percent. Results are presented to show that the developed methodology produces a smaller number of required MPAR units using less time than a comparable method presented in the literature. A goodness measure of the method is presented with respect to a lower bound for over 1000 test cases. Results for the United States Northern Command MPAR instance of this problem are presented to provide full and partial coverage of the Continental United States and 25 key cities of interest. The methodology developed in this thesis can be used to provide minimal cost surveillance recommendations over key areas or events, placement of communications resources, or other limited range resources.

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